

# Modified first-order Hořava-Lifshitz gravity: Hamiltonian analysis of the general theory and accelerating FRW cosmology in power-law $F(R)$ model

Sante Carloni<sup>1</sup>, Masud Chaichian<sup>2,3</sup>, Shin'ichi Nojiri<sup>4</sup>,  
Sergei D. Odintsov<sup>1,5\*</sup>, Markku Oksanen<sup>2</sup>, and Anca Tureanu<sup>2,3</sup>

<sup>1</sup> *Institut de Ciències de l'Espai (IEEC-CSIC), Campus UAB,*

*Facultat de Ciències, Torre C5-Par-2a pl, E-08193 Bellaterra (Barcelona), Spain*

<sup>2</sup> *Department of Physics, University of Helsinki, P.O. Box 64, FI-00014 Helsinki, Finland*

<sup>3</sup> *Helsinki Institute of Physics, P.O. Box 64, FI-00014 Helsinki, Finland*

<sup>4</sup> *Department of Physics, Nagoya University, Nagoya 464-8602, Japan*

<sup>5</sup> *Institució Catalana de Recerca i Estudis Avançats (ICREA), Barcelona*

We propose the most general modified first-order Hořava-Lifshitz gravity, whose action does not contain time derivatives higher than the second order. The Hamiltonian structure of this theory is studied in all the details in the case of the spatially-flat FRW space-time, demonstrating many of the features of the general theory. It is shown that, with some plausible assumptions, including the projectability of the lapse function, this model is consistent. As a large class of such theories, the modified Hořava-Lifshitz  $F(R)$  gravity is introduced. The study of its ultraviolet properties shows that its  $z = 3$  version seems to be renormalizable in the same way as the original Hořava-Lifshitz proposal. The Hamiltonian analysis of the modified Hořava-Lifshitz  $F(R)$  gravity shows that it is in general a consistent theory. The  $F(R)$  gravity action is also studied in the fixed-gauge form, where the appearance of a scalar field is particularly illustrative. Then the spatially-flat FRW cosmology for this  $F(R)$  gravity is investigated. It is shown that a special choice of parameters for this theory leads to the same equations of motion as in the case of traditional  $F(R)$  gravity. Nevertheless, the cosmological structure of the modified Hořava-Lifshitz  $F(R)$  gravity turns out to be much richer than for its traditional counterpart. The emergence of multiple de Sitter solutions indicates to the possibility of unification of early-time inflation with late-time acceleration within the same model. Power-law  $F(R)$  theories are also investigated in detail. It is analytically shown that they have a quite rich cosmological structure: early/late-time cosmic acceleration of quintessence, as well as of phantom types. Also it is demonstrated that all the four known types of finite-time future singularities may occur in the power-law Hořava-Lifshitz  $F(R)$  gravity. Finally, a covariant proposal for (renormalizable)  $F(R)$  gravity within the Hořava-Lifshitz spirit is presented.

PACS numbers: 11.10.Ef, 95.36.+x, 98.80.Cq, 04.50.Kd, 11.25.-w

## I. INTRODUCTION

Recently, it has become clear that our universe has not only undergone the period of early-time accelerated expansion (inflation), but also is currently in the so-called late-time accelerating epoch (dark energy era). An extremely powerful way to describe the early-time inflation and the late-time acceleration in a unified manner is modified gravity. This approach does not require the introduction of new dark components like inflaton and dark energy. The unified description of inflation and dark energy is achieved by modifying the gravitational action at the very early universe as well as at the very late times (for a review of such models, see [1]). A number of viable modified gravity theories has been suggested. Despite some indications to possible connection with string/M-theory [2], such theories remain to be mainly phenomenological. It is a challenge to investigate their origin from some (not yet constructed) fundamental quantum gravity theory.

Among the recent attempts to construct a consistent theory of quantum gravity much attention has been paid to the quite remarkable Hořava-Lifshitz quantum gravity [3], which appears to be power-counting renormalizable in four dimensions. In this theory the local Lorentz invariance is abandoned, but it is restored as an approximate symmetry at low energies. Despite its partial success as a candidate for fundamental theory of gravity, there are a number of unresolved problems related to the detailed balance and projectability conditions, consistency, its general relativity

---

\* Also at Tomsk State Pedagogical University

(GR) limit, realistic cosmological applications, the relation to other modified gravities, etc. Due to the fact that its spatially-flat FRW cosmology [4] is almost the same as in GR, it is difficult to obtain a unified description of the early-time inflation with the late-time acceleration in the standard Hořava-Lifshitz gravity.

Recently the modified Hořava-Lifshitz  $F(R)$  gravity has been proposed [5]. Such a modification may be easily related with the traditional modified gravity approach, but turns out to be much richer in terms of the possible cosmological solutions. For instance, the unification of inflation with dark energy seems to be possible in such Hořava-Lifshitz gravity due to the presence of multiple de Sitter solutions. Moreover, on the one hand, there is the hope that the generalization of Hořava-Lifshitz gravity may lead to new classes of renormalizable quantum gravity. On the other hand, one may hope to formulate the dynamical scenario for the Lorentz symmetry violation/restoration, caused by the expansion of the universe, in terms of such generalized theory.

In the present work (section II) we propose the most general modified first-order Hořava-Lifshitz-like theory, without higher derivative terms which are normally responsible for the presence of ghosts. The general form of the action in the spatially-flat FRW space-time is found, and the Hamiltonian structure of the action is analyzed in section III.

As a specific example of such a first-order action we introduce the modified Hořava-Lifshitz  $F(R)$  theory which is more general than the model of ref. [5]. Nevertheless, its spatially-flat FRW cosmology turns out to be the same as for the model [5] (this is not the case for BH solutions, etc). Therefore it also coincides with the conventional  $F(R)$  spatially-flat cosmology for a specific choice of the parameters. The ultraviolet structure of the new Hořava-Lifshitz  $F(R)$  gravity is carefully investigated. It is shown that such models can have very nice ultraviolet behaviour at  $z = 2$ . Moreover, for  $z = 3$  a big class of renormalizable models is suggested (section II). The Hamiltonian analysis of the modified Hořava-Lifshitz  $F(R)$  gravity is presented in section IV. The fixed gauge modified Hořava-Lifshitz  $F(R)$  gravity is analyzed in section V.

Section VI is devoted to the investigation of spatially-flat FRW cosmology for power-law  $F(R)$  gravity. The general equation for the de Sitter solutions is obtained. It acquires an extremely simple form for a special choice of parameters, when de Sitter solutions are roots of the equation  $F = 0$ . The existence of multiple de Sitter solutions indicates the principal possibility of attaining the unification of the early-time inflation with the late-time acceleration in the modified Hořava-Lifshitz  $F(R)$  gravity. The reconstruction technique is developed for the study of analytical and accelerating FRW cosmologies in power-law models. A number of explicit analytical solutions are presented. It is shown by explicit examples that some of the quintessence/phantom-like cosmologies may develop the future finite-time singularity of all the known four types, precisely in the same way as for traditional dark energy models. The possible curing of such singularities could be achieved in a similar way as in the case of traditional modified gravity. Some remarks about small corrections to the Newton law are made in section VII. A summary and outlook are given in the last section VIII. In the appendix A we propose a covariant  $F(R)$  gravity that is quite similar to the corresponding Hořava-Lifshitz version but remains to be a covariant theory. It seems that it could also be made renormalizable.

## II. GENERAL ACTION FOR HOŘAVA-LIFSHITZ-LIKE GRAVITY AND RENORMALIZABILITY

In this section we propose the essentially most general Hořava-Lifshitz-like gravity action, which does not contain derivatives with respect to the time coordinate higher than the second order. Its ultraviolet properties are discussed.

By using the ADM decomposition [6] (for reviews and mathematical background, see [7]), one can write the metric of space-time in the following form:

$$ds^2 = -N^2 dt^2 + g_{ij}^{(3)} (dx^i + N^i dt) (dx^j + N^j dt), \quad i, j = 1, 2, 3. \quad (1)$$

Here  $N$  is called the lapse variable and  $N^i$  is the shift 3-vector. Then the scalar curvature  $R$  has the following form:

$$R = K^{ij} K_{ij} - K^2 + R^{(3)} + 2\nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu). \quad (2)$$

Here  $R^{(3)}$  is the three-dimensional scalar curvature defined by the metric  $g_{ij}^{(3)}$  and  $K_{ij}$  is the extrinsic curvature defined by

$$K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij}^{(3)} - \nabla_i^{(3)} N_j - \nabla_j^{(3)} N_i \right), \quad K = K^i_i. \quad (3)$$

$n^\mu$  is the unit vector perpendicular to the three-dimensional space-like hypersurface  $\Sigma_t$  defined by  $t = \text{constant}$  and  $\nabla_i^{(3)}$  is the covariant derivative on the hypersurface  $\Sigma_t$ . From the determinant of the metric (1) one obtains  $\sqrt{-g} = \sqrt{g^{(3)}} N$ .

For general Hořava-Lifshitz-like gravity models, we do not require the full diffeomorphism-invariance, but only invariance under “foliation-preserving” diffeomorphisms:

$$\delta x^i = \zeta^i(t, \mathbf{x}), \quad \delta t = f(t). \quad (4)$$

Therefore, there are many invariants or covariant quantities made from the metric, in particular  $K, K_{ij}, \nabla_i^{(3)} K_{jk}, \dots, \nabla_{i_1}^{(3)} \nabla_{i_2}^{(3)} \dots \nabla_{i_n}^{(3)} K_{jk}, \dots, R^{(3)}, R_{ij}^{(3)}, R_{ijkl}^{(3)}, \nabla_i^{(3)} R_{jklm}^{(3)}, \dots, \nabla_{i_1}^{(3)} \nabla_{i_2}^{(3)} \dots \nabla_{i_n}^{(3)} R_{jklm}^{(3)}, \dots, \nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu), \dots$ , etc. Then the general consistent action composed of invariants that are constructed from such covariant quantities,

$$S_{\text{gHL}} = \int d^4x \sqrt{g^{(3)}} N F \left( g_{ij}^{(3)}, K, K_{ij}, \nabla_i^{(3)} K_{jk}, \dots, \nabla_{i_1}^{(3)} \nabla_{i_2}^{(3)} \dots \nabla_{i_n}^{(3)} K_{jk}, \dots, R^{(3)}, R_{ij}^{(3)}, R_{ijkl}^{(3)}, \nabla_i^{(3)} R_{jklm}^{(3)}, \dots, \nabla_{i_1}^{(3)} \nabla_{i_2}^{(3)} \dots \nabla_{i_n}^{(3)} R_{jklm}^{(3)}, \dots, \nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu) \right), \quad (5)$$

could be a rather general action for the generalized Hořava-Lifshitz gravity. Note that one can also include the (cosmological) constant in the above action. Here it has been assumed that the action does not contain derivatives higher than the second order with respect to the time coordinate  $t$ . In the usual  $F(R)$  gravity, there appears the extra scalar mode, since the equations given by the variation over the metric tensor contain the fourth derivative. By assuming that the action does not contain derivatives higher than the second order with respect to the time coordinate  $t$ , we can avoid more extra modes in addition to the only one scalar mode which appears in the usual  $F(R)$  gravity. For example, if we consider the action containing the terms like

$$(\nabla^\mu \nabla_\mu)^{n+1} R^{(3)}, \quad (\nabla^\rho \nabla_\rho)^n \nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu), \quad (6)$$

the equations given by the variation over the metric tensor contain the fifth or higher derivatives (for a review of Hamiltonian structure of higher derivative modified gravity, see [8]). If we define new fields recursively

$$\chi_R^{(m+1)} = \nabla^\mu \nabla_\mu \chi_R^{(m)}, \quad \chi_R^{(0)} = R^{(3)}, \quad \chi_n^{(m+1)} = \nabla^\mu \nabla_\mu \chi_n^{(m)}, \quad \chi_n^{(0)} = \nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu), \quad (7)$$

the equations can be rewritten so that only second derivatives appear. The scalar fields in (7), however, often become ghost fields that generate states of negative norm. Thus, we only consider actions of the form given by (5) in this paper.

In the Hořava-Lifshitz-type gravity, we assume that  $N$  can only depend on the time coordinate  $t$ , which is called the *projectability condition*. The reason is that the Hořava-Lifshitz gravity does not have the full diffeomorphism-invariance, but is invariant only under the foliation-preserving diffeomorphisms (4). If  $N$  depended on the spatial coordinates, we could not fix  $N$  to be unity ( $N = 1$ ) by using the foliation-preserving diffeomorphisms. Moreover, there are strong reasons to suspect that the non-projectable version of the Hořava-Lifshitz gravity is generally inconsistent [9]. Therefore we prefer to assume that  $N$  is projectable.

In the FRW space-time with the flat spatial part and the non-trivial lapse  $N(t)$ ,

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2, \quad (8)$$

we find

$$\begin{aligned} \Gamma_{00}^0 &= \frac{\dot{N}}{N}, \quad \Gamma_{ij}^0 = \frac{a^2 H}{N^2} \delta_{ij}, \quad \Gamma_{j0}^i = H \delta_j^i \quad \text{other } \Gamma_{\nu\rho}^\mu = 0, \\ K_{ij} &= \frac{a^2 H}{N} \delta_{ij}, \quad \nabla_i^{(3)} = 0, \quad R_{ijkl}^{(3)} = 0, \quad \nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu) = \frac{3}{a^3 N} \frac{d}{dt} \left( \frac{a^3 H}{N} \right), \end{aligned} \quad (9)$$

where  $H = \frac{\dot{a}}{a}$  is the Hubble parameter. Then one gets

$$\begin{aligned} g_{ij}^{(3)} &= a^2 \delta_{ij}, \quad K = \frac{3H}{N}, \quad K_{ij} K^{ij} = 3 \left( \frac{H}{N} \right)^2, \quad \nabla_i^{(3)} K_{jk} = \dots = \nabla_{i_1}^{(3)} \nabla_{i_2}^{(3)} \dots \nabla_{i_n}^{(3)} K_{jk} = \dots = 0, \\ R^{(3)} &= R_{ij}^{(3)} = R_{ijkl}^{(3)} = \nabla_i^{(3)} R_{jklm} = \dots = \nabla_{i_1}^{(3)} \nabla_{i_2}^{(3)} \dots \nabla_{i_n}^{(3)} R_{jklm} = \dots = 0, \end{aligned} \quad (10)$$

and since  $F$  must be a scalar under the spatial rotation, the action (5) reduces to

$$S_{\text{gHL}} = \int d^4x \sqrt{g^{(3)}} N F \left( \frac{H}{N}, \frac{3}{a^3 N} \frac{d}{dt} \left( \frac{a^3 H}{N} \right) \right). \quad (11)$$

Therefore, if we consider the FRW cosmology, the function  $F$  should depend on only two variables,  $\frac{H}{N}$  and  $\frac{3}{a^3 N} \frac{d}{dt} \left( \frac{a^3 H}{N} \right)$ .

As a specific example of the above general theory, we may consider the following modified Hořava-Lifshitz  $F(R)$  gravity, whose action is given by

$$S_{F(\tilde{R})} = \frac{1}{2\kappa^2} \int d^4x \sqrt{g^{(3)}} N F(\tilde{R}), \quad \tilde{R} \equiv K^{ij} K_{ij} - \lambda K^2 + 2\mu \nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu) - \mathcal{L}_R^{(3)}(g_{ij}^{(3)}). \quad (12)$$

Here  $\lambda$  and  $\mu$  are constants and  $\mathcal{L}_R^{(3)}$  is a function of the three-dimensional metric  $g_{ij}^{(3)}$  and the covariant derivatives  $\nabla_i^{(3)}$  defined by this metric. Note that this action (12) is more general than the one introduced in ref. [5] due to the presence of the last term in  $\tilde{R}$ . We normalize  $F(\tilde{R})$  or redefine  $\kappa^2$  so that

$$F'(0) = 1. \quad (13)$$

In [3],  $\mathcal{L}_R^{(3)}$  is chosen to be

$$\mathcal{L}_R^{(3)}(g_{ij}^{(3)}) = E^{ij} \mathcal{G}_{ijkl} E^{kl}, \quad (14)$$

where  $\mathcal{G}_{ijkl}$  is the “generalized De Witt metric” or “super-metric” (“metric of the space of metric”),

$$\mathcal{G}^{ijkl} = \frac{1}{2} \left( g^{(3)ik} g^{(3)jl} + g^{(3)il} g^{(3)jk} \right) - \lambda g^{(3)ij} g^{(3)kl}, \quad (15)$$

defined on the three-dimensional hypersurface  $\Sigma_t$ .  $E^{ij}$  can be defined by the so called *detailed balance condition* by using an action  $W[g_{kl}^{(3)}]$  on the hypersurface  $\Sigma_t$

$$\sqrt{g^{(3)}} E^{ij} = \frac{\delta W[g_{kl}^{(3)}]}{\delta g_{ij}^{(3)}}, \quad (16)$$

and the inverse of  $\mathcal{G}^{ijkl}$  is written as

$$\mathcal{G}_{ijkl} = \frac{1}{2} \left( g_{ik}^{(3)} g_{jl}^{(3)} + g_{il}^{(3)} g_{jk}^{(3)} \right) - \tilde{\lambda} g_{ij}^{(3)} g_{kl}^{(3)}, \quad \tilde{\lambda} = \frac{\lambda}{3\lambda - 1}. \quad (17)$$

The action  $W[g_{kl}^{(3)}]$  is assumed to be defined by the metric and the covariant derivatives on the hypersurface  $\Sigma_t$ . There is an anisotropy between space and time in the Hořava-Lifshitz gravity. In the ultraviolet (high energy) region, the time coordinate and the spatial coordinates are assumed to behave as

$$\mathbf{x} \rightarrow b\mathbf{x}, \quad t \rightarrow b^z t, \quad z = 2, 3, \dots, \quad (18)$$

under the scale transformation. In [3],  $W[g_{kl}^{(3)}]$  is explicitly given for the case  $z = 2$ ,

$$W = \frac{1}{\kappa_W^2} \int d^3\mathbf{x} \sqrt{g^{(3)}} \left( R^{(3)} - 2\Lambda_W \right), \quad (19)$$

and for the case  $z = 3$ ,

$$W = \frac{1}{w^2} \int_{\Sigma_t} \omega_3(\Gamma), \quad (20)$$

where

$$\omega_3(\Gamma) = \text{Tr} \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) \equiv \varepsilon^{ijk} \left( \Gamma_{il}^m \partial_j \Gamma_{km}^l + \frac{2}{3} \Gamma_{il}^n \Gamma_{jm}^l \Gamma_{kn}^m \right) d^3\mathbf{x}. \quad (21)$$

Here  $\kappa_W$  in (19) is a coupling constant of dimension  $-1/2$  and  $w^2$  in (20) is a dimensionless coupling constant. A general  $E^{ij}$  consist of all contributions to  $W$  up to the chosen value  $z$ . The original motivation for the detailed balance condition is its ability to simplify the quantum behaviour and renormalization properties of theories that respect it. Otherwise there is no a priori physical reason to restrict  $\mathcal{L}_R^{(3)}$  to be defined by (14). In the following we abandon the detailed balance condition and consider  $\mathcal{L}_R^{(3)}$  to have a more general form, since it is not always relevant even for the renomalizability problem.

We now investigate the renormalizability and the unitarity of the model (12). For this purpose, by introducing an auxiliary field  $A$ , we rewrite the action (12) in the following form:

$$S_{F(\tilde{R})} = \frac{1}{2\kappa^2} \int d^4x \sqrt{g^{(3)}} N \left\{ F'(A)(\tilde{R} - A) + F(A) \right\}. \quad (22)$$

For simplicity, the following gauge condition is used:

$$N = 1, \quad N^i = 0. \quad (23)$$

Then one finds

$$\begin{aligned} \Gamma_{ij}^0 &= -\frac{1}{2} \dot{g}_{ij}^{(3)}, \quad \Gamma_{j0}^i = \Gamma_{0j}^i = \frac{1}{2} g^{(3)ik} \dot{g}_{kj}^{(3)}, \quad \Gamma_{jk}^i = \Gamma_{jk}^{(3)i} \equiv \frac{1}{2} g^{(3)il} \left( g_{lk,j}^{(3)} + g_{jl,k}^{(3)} - g_{jk,l}^{(3)} \right), \\ \text{other components of } \Gamma_{\nu\rho}^\mu &= 0, \end{aligned} \quad (24)$$

and therefore

$$(n^\mu) = (1, 0, 0, 0), \quad K_{ij} = \frac{1}{2} \dot{g}_{ij}^{(3)}, \quad \nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu) = \frac{1}{2} \partial_0 \left( g^{(3)ij} \dot{g}_{ij}^{(3)} \right) + \frac{1}{4} \left( g^{(3)ij} \dot{g}_{ij}^{(3)} \right)^2. \quad (25)$$

We define a new field by

$$\varphi \equiv \frac{1}{3} \ln F'(A), \quad (26)$$

which can be algebraically solved as  $A = A(\varphi)$ , so that

$$\varphi = \frac{1}{3} \ln F'(A(\varphi)) \quad \Leftrightarrow \quad F'(A(\varphi)) = e^{3\varphi}. \quad (27)$$

The spatial metric is redefined as

$$g_{ij}^{(3)} = e^{-\varphi} \bar{g}_{ij}^{(3)}. \quad (28)$$

Then the action (22) has the following form:

$$\begin{aligned} S_{F(\tilde{R})} &= \frac{1}{2\kappa^2} \int d^4x \sqrt{\bar{g}^{(3)}} \left\{ \frac{1}{4} \bar{g}^{(3)ij} \bar{g}^{(3)kl} \dot{\bar{g}}_{ik}^{(3)} \dot{\bar{g}}_{jl}^{(3)} - \frac{\lambda}{4} \left( \bar{g}^{(3)ij} \dot{\bar{g}}_{ij}^{(3)} \right)^2 \right. \\ &\quad \left. + \left( -\frac{1}{2} + \frac{3\lambda}{2} - \frac{3\mu}{2} \right) \bar{g}^{(3)ij} \dot{\bar{g}}_{ij}^{(3)} \dot{\varphi} + \left( \frac{3}{4} - \frac{9\lambda}{4} + \frac{9\mu}{2} \right) \dot{\varphi}^2 + \bar{\mathcal{L}}_R^{(3)}(\bar{g}_{ij}^{(3)}, \varphi) - V(\varphi) \right\}. \end{aligned} \quad (29)$$

Here

$$\bar{\mathcal{L}}_R^{(3)}(\bar{g}_{ij}^{(3)}, \varphi) \equiv \mathcal{L}_R^{(3)}(e^{-\varphi} \bar{g}_{ij}^{(3)}), \quad V(\varphi) \equiv A(\varphi) F'(A(\varphi)) - F(A(\varphi)). \quad (30)$$

If we insert  $\varphi = 1$  into the action (29), the standard Hořava-Lifshitz gravity emerges. On the other hand, if we choose

$$\mu = \lambda - \frac{1}{3}, \quad (31)$$

$\dot{\varphi}$  decouples with  $\dot{g}_{ij}^{(3)}$ . When the decoupling (31) is assumed and

$$\lambda > \frac{1}{3}, \quad (32)$$

$\varphi$  becomes canonical and the theory becomes unitary. In the case

$$\lambda = \frac{1}{3}, \quad (33)$$

the  $\dot{\varphi}^2$  term vanishes and therefore  $\varphi$  becomes non-dynamical, i.e. an auxiliary field. Eq. (31) tells that  $\mu = 0$  when  $\lambda = 1/3$ .

In order to clarify the renormalizability issue, we need to explicitly construct  $\mathcal{L}_R^{(3)}(g_{ij}^{(3)})$  in (12). As a model corresponding to  $z = 2$  in (18), which is still not renormalizable, we may propose

$$\mathcal{L}_R^{(3)}(g_{ij}^{(3)}) = c_2 \left( R^{(3)ij} R_{ij}^{(3)} + \alpha \left( R^{(3)} \right)^2 \right), \quad (34)$$

where  $c_2$  and  $\alpha$  are constants. Since the action (34) induces the higher derivative terms to contribute to the propagators and therefore the propagators behave as  $1/|\mathbf{k}|^4$  in the high energy region, the ultraviolet behavior is improved, although the theory still is not renormalizable.

By the scale transformation (28), the curvatures are transformed as

$$\begin{aligned} R_{ij}^{(3)} &= \bar{R}_{ij}^{(3)} + \frac{1}{2} \left( \bar{\nabla}_i^{(3)} \bar{\nabla}_j^{(3)} \varphi + \bar{g}_{ij}^{(3)} \bar{\Delta}^{(3)} \varphi \right) + \frac{1}{4} \left( \bar{\nabla}_i^{(3)} \varphi \bar{\nabla}_j^{(3)} \varphi - \bar{g}_{ij}^{(3)} \bar{g}^{(3)kl} \bar{\nabla}_k^{(3)} \varphi \bar{\nabla}_l^{(3)} \varphi \right), \\ R^{(3)} &= e^\varphi \left( \bar{R}^{(3)} + 2 \bar{\Delta}^{(3)} \varphi - \frac{1}{2} \bar{g}^{(3)kl} \bar{\nabla}_k^{(3)} \varphi \bar{\nabla}_l^{(3)} \varphi \right). \end{aligned} \quad (35)$$

Here  $\bar{R}_{ij}^{(3)}$ ,  $\bar{R}^{(3)}$ ,  $\bar{\nabla}_i^{(3)}$ , and  $\bar{\Delta}^{(3)}$  are the Ricci curvature, the scalar curvature, the covariant derivative, and the Laplacian defined by the metric  $\bar{g}_{ij}^{(3)}$ , respectively. Then if we consider the perturbation from the flat background, where  $\varphi \sim 0$  due to (13),

$$\bar{g}_{ij}^{(3)} = \delta_{ij} + \bar{h}_{ij}^{(3)}, \quad |\bar{h}_{ij}^{(3)}|, |\varphi| \ll 1, \quad (36)$$

we find

$$\begin{aligned} &\int d^4x \sqrt{g^{(3)}} N \bar{\mathcal{L}}_R^{(3)}(\bar{g}_{ij}^{(3)}, \varphi) \\ &= \int d^4x \left[ \frac{1}{4} \left\{ \partial_i \partial^k \bar{h}_{kj}^{(3)} + \partial_j \partial^k \bar{h}_{ki}^{(3)} - \partial_i \partial_j \bar{h}_k^{(3)k} - \Delta \bar{h}_{ij}^{(3)} \right\} \left\{ \partial^i \partial^l \bar{h}_l^{(3)j} + \partial^j \partial^l \bar{h}_l^{(3)i} - \partial^i \partial^j \bar{h}_l^{(3)l} - \Delta \bar{h}^{(3)ij} \right\} \right. \\ &\quad \left. + \alpha \left\{ \partial_i \partial^j \bar{h}_{ij}^{(3)} - \Delta \bar{h}_k^{(3)k} \right\}^2 + \left( -\frac{1}{2} + 4\alpha \right) \left\{ \partial_i \partial^j \bar{h}_{ij}^{(3)} - \Delta \bar{h}_k^{(3)k} \right\} \Delta \varphi + \left( \frac{1}{2} + 4\alpha \right) (\Delta \varphi)^2 \right]. \end{aligned} \quad (37)$$

Therefore if one chooses

$$\alpha = \frac{1}{8}, \quad (38)$$

$\varphi$  decouples with  $\bar{h}_{ij}^{(3)}$ . Eq. (37) shows that the propagators of  $\varphi$  and  $\bar{h}_{ij}^{(3)}$  behave as  $1/|\mathbf{k}|^4$  in the high energy region, so that the ultraviolet behavior is improved.

Similarly, a model corresponding to  $z = 3$  in (18), which could be power-counting renormalizable, can be obtained by choosing

$$\mathcal{L}_R^{(3)}(g_{ij}^{(3)}) = c_3 \left( \bar{\nabla}^{(3)k} R^{(3)ij} \bar{\nabla}_k^{(3)} R_{ij}^{(3)} + \frac{1}{8} \bar{\nabla}^{(3)k} R^{(3)} \bar{\nabla}_k^{(3)} R^{(3)} \right). \quad (39)$$

For the  $z = 3$  model, the dimension of  $\varphi$  vanishes and therefore all the interactions in  $\bar{\mathcal{L}}_R^{(3)}(\bar{g}_{ij}^{(3)}, \varphi)$  and  $V(\varphi)$  in (29) become power-counting renormalizable. The propagators of  $\varphi$  and  $\bar{h}_{ij}^{(3)}$  behave as  $1/|\mathbf{k}|^6$  in the high energy region, so that the ultraviolet behavior is improved to be renormalizable.

We have shown that by requiring (31) and (38), the scalar field decouples with the gravity modes in the Einstein frame. The decoupling itself is not directly related with the renormalizability, but the decoupling makes it much easier to discuss the renormalizability of the model. The choice (39) for  $\mathcal{L}_R^{(3)}$  gives the renormalizable model. The renormalizability does not essentially depend on the functional form of  $F(R)$ .

### III. HAMILTONIAN ANALYSIS OF THE GENERAL ACTION IN THE FRW SPACE-TIME

Let us analyze the proposed general action (11) of the FRW space-time (8) with the flat spatial part and the non-trivial lapse  $N = N(t)$ . Introducing four auxiliary variables  $\alpha, A, \beta, B$  enables us to write the action (11) as

$$S_{\text{gHL}} = \int d^4x \sqrt{g^{(3)}} N \left[ \alpha \left( A - \frac{H}{N} \right) + \beta \left( B - \frac{3}{a^3 N} \frac{d}{dt} \left( \frac{a^3 H}{N} \right) \right) + F(A, B) \right]. \quad (40)$$

The variations of the action (40) with respect to  $\alpha$  and  $\beta$  yield

$$A = \frac{H}{N} \quad \text{and} \quad B = \frac{3}{a^3 N} \frac{d}{dt} \left( \frac{a^3 H}{N} \right), \quad (41)$$

respectively. Integration by parts permits the removal of the second-order time derivative of  $a$  and the time derivative of  $N$ , assuming the boundary terms vanish, but with the price that  $\beta$  becomes a dynamical variable. Thus the action (40) can be written as

$$S_{\text{gHL}} = \int d^4x \sqrt{g^{(3)}} N \left[ \alpha \left( A - \frac{H}{N} \right) + \beta B + \frac{3\dot{\beta}H}{N^2} + F(A, B) \right], \quad (42)$$

The action (42) is equivalent to (40) and consequently to the original action (11). The advantage of the action (42) over (11) is the simpler dependence on the variables  $a$  and  $N$ , which will be crucially important in the following Hamiltonian analysis.

For the Hamiltonian analysis of constrained systems and their quantization we refer to the monographs [10–13].

In the Hamiltonian formalism the generalized coordinates  $g_{ij}^{(3)}$ ,  $N$ ,  $\alpha$ ,  $A$ ,  $\beta$  and  $B$  of the action (42) have the canonically conjugated momenta  $\pi^{ij}$ ,  $\pi_N$ ,  $\pi_\alpha$ ,  $\pi_A$ ,  $\pi_\beta$  and  $\pi_B$ , respectively. We consider  $N$  to be projectable,  $N = N(t)$ , and therefore also the momentum  $\pi_N = \pi_N(t)$  is constant on the hypersurface  $\Sigma_t$  for each  $t$ . The Poisson brackets are postulated in the form (equal time  $t$  is understood)

$$\begin{aligned} \{g_{ij}^{(3)}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} &= \frac{1}{2} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(\mathbf{x} - \mathbf{y}), \quad \{N, \pi_N\} = 1, \\ \{\alpha(\mathbf{x}), \pi_\alpha(\mathbf{y})\} &= \delta(\mathbf{x} - \mathbf{y}), \quad \{A(\mathbf{x}), \pi_A(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \\ \{\beta(\mathbf{x}), \pi_\beta(\mathbf{y})\} &= \delta(\mathbf{x} - \mathbf{y}), \quad \{B(\mathbf{x}), \pi_B(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (43)$$

with all the other Poisson brackets vanishing. We are considering the FRW metric (8) with the flat spatial part  $g_{ij}^{(3)} = a^2 \delta_{ij}$  and therefore the Poisson bracket for the scale factor  $a$  and the momenta conjugate to the 3-metric takes the form

$$\int d^3\mathbf{y} \{a, \pi^{ij}(\mathbf{y})\} = \frac{\delta^{ij}}{2a}. \quad (44)$$

Let us find the momenta and the primary constraints. The action (42) does not depend on the time derivative of  $N$ ,  $\alpha$ ,  $A$  or  $B$ . Thus we have the primary constraints

$$\Phi_1 \equiv \pi_N \approx 0, \quad \Phi_2(\mathbf{x}) \equiv \pi_\alpha(\mathbf{x}) \approx 0, \quad \Phi_3(\mathbf{x}) \equiv \pi_A(\mathbf{x}) \approx 0, \quad \Phi_4(\mathbf{x}) \equiv \pi_B(\mathbf{x}) \approx 0. \quad (45)$$

The momenta conjugated to  $\beta$  and  $g_{ij}^{(3)}$  are

$$\pi_\beta = \frac{\delta S_{\text{gHL}}}{\delta \dot{\beta}} = \frac{3a^3 H}{N}, \quad (46)$$

$$\pi^{ij} = \frac{\delta S_{\text{gHL}}}{\delta \dot{g}_{ij}^{(3)}} = \frac{a}{6} \left( -\alpha + \frac{3\dot{\beta}}{N} \right) \delta^{ij}, \quad (47)$$

respectively. The “velocities”  $\dot{\beta}$  and  $\dot{g}_{ij}^{(3)}$  can be solved in terms of the canonical variables, so there are no more primary constraints.

Then we define the Hamiltonian

$$H = \int d^3\mathbf{x} \left( \pi^{ij} \dot{g}_{ij}^{(3)} + \pi_\beta \dot{\beta} \right) - L = \int d^3\mathbf{x} N \mathcal{H}, \quad (48)$$

where the Lagrangian  $L$  is defined by (42),  $S_{\text{gHL}} = \int dt L$ , and the so-called Hamiltonian constraint is found to be

$$\mathcal{H} = \frac{\pi_\beta}{3} \left( \frac{2}{a} \sum_{i=1}^3 \pi^{ii} + \alpha \right) - a^3 (\alpha A + \beta B + F(A, B)). \quad (49)$$

The primary constraints (45) can be included into the Hamiltonian (48) by using the Lagrange multipliers  $\lambda_k$ ,  $k = 1, 2, 3, 4$ . We define the total Hamiltonian by

$$H_T = H + \lambda_1 \Phi_1 + \sum_{n=2}^4 \int d^3\mathbf{x} \lambda_n(\mathbf{x}) \Phi_n(\mathbf{x}). \quad (50)$$

Note that there is no space integral over the product  $\lambda_1 \Phi_1 = \lambda_1 \pi_N$ , since these variables depend only on the time coordinate  $t$ .

The consistency of the system requires that every constraint has to be preserved under time evolution. Since the Poisson brackets of the primary constraints (45) are zero,

$$\{\Phi_k, \Phi_l\} = 0, \quad k, l \in \{1, 2, 3, 4\}, \quad (51)$$

the time evolution of the primary constraints is determined by the Hamiltonian  $H$  alone

$$\dot{\Phi}_k = \{\Phi_k, H_T\} = \{\Phi_k, H\}, \quad k = 1, 2, 3, 4. \quad (52)$$

Thus the following time derivatives of the primary constraints have to vanish:

$$\begin{aligned} \dot{\Phi}_1 &= \dot{\pi}_N = \{\pi_N, H\} = - \int d^3\mathbf{x} \mathcal{H}, \\ \dot{\Phi}_2 &= \dot{\pi}_\alpha = \{\pi_\alpha, H\} = N \left( -\frac{\pi_\beta}{3} + a^3 A \right), \\ \dot{\Phi}_3 &= \dot{\pi}_A = \{\pi_A, H\} = N a^3 \left( \alpha + \frac{\partial F(A, B)}{\partial A} \right), \\ \dot{\Phi}_4 &= \dot{\pi}_B = \{\pi_B, H\} = N a^3 \left( \beta + \frac{\partial F(A, B)}{\partial B} \right). \end{aligned} \quad (53)$$

Since none of these expressions (53) vanish due to the primary constraints (45), we must impose the secondary constraints:

$$\begin{aligned} \Phi_0 &\equiv \int d^3\mathbf{x} \mathcal{H} \approx 0, \\ \Phi_5(\mathbf{x}) &\equiv -\frac{\pi_\beta}{3} + a^3 A \approx 0, \\ \Phi_6(\mathbf{x}) &\equiv \alpha + \frac{\partial F(A, B)}{\partial A} \approx 0, \\ \Phi_7(\mathbf{x}) &\equiv \beta + \frac{\partial F(A, B)}{\partial B} \approx 0. \end{aligned} \quad (54)$$

Here the position argument  $\mathbf{x}$  has been omitted in the right-hand side of the local constraints. Note that neither  $N$  or  $a$  can be constrained to vanish, since they are the essential physical quantities in this theory. Here the actual Hamiltonian constraint  $\Phi_0$  is global due to the projectability condition,  $N = N(t)$ . Note that the Hamiltonian (48) is simply this constraint multiplied by  $N$ , i.e.

$$H = N \Phi_0. \quad (55)$$



Also the secondary constraints (54) have to be preserved under time evolution. The time evolution of the secondary constraints is

$$\dot{\Phi}_m = \{\Phi_m, H_T\} = N\{\Phi_m, \Phi_0\} + \sum_{n=2}^4 \int d^3\mathbf{y} \lambda_n(\mathbf{y}) \{\Phi_m, \Phi_n(\mathbf{y})\}, \quad m = 0, 5, 6, 7, \quad (56)$$

where we have used (55) and the fact that none of the constraints  $\Phi_j, j = 1, 2, \dots, 7, 0$  depend on the lapse  $N$ , and that the secondary constraints (54) do not depend on  $\pi_N$ . For the global Hamiltonian constraint  $\Phi_0$  we find the following Poisson brackets with the primary constraints (45)

$$\{\Phi_0, \Phi_2(\mathbf{x})\} = -\Phi_5(\mathbf{x}), \quad \{\Phi_0, \Phi_3(\mathbf{x})\} = -a^3\Phi_6(\mathbf{x}), \quad \{\Phi_0, \Phi_4(\mathbf{x})\} = -a^3\Phi_7(\mathbf{x}), \quad (57)$$

which all vanish due to the other secondary constraints. Thus, according to (56) and (57), the Hamiltonian constraint  $\Phi_0$  is preserved under time evolution,  $\dot{\Phi}_0 \approx 0$ . For the secondary constraint  $\Phi_5$  we obtain the non-vanishing Poisson brackets with the primary constraints (45) and the Hamiltonian constraint  $\Phi_0$

$$\{\Phi_5(\mathbf{x}), \Phi_3(\mathbf{y})\} = a^3\delta(\mathbf{x} - \mathbf{y}), \quad \{\Phi_5, \Phi_0(\mathbf{x})\} = -\frac{a^3B}{3} + 3\pi_\beta A. \quad (58)$$

For the next secondary constraint  $\Phi_6$  we obtain the non-vanishing Poisson brackets

$$\begin{aligned} \{\Phi_6(\mathbf{x}), \Phi_2(\mathbf{y})\} &= \delta(\mathbf{x} - \mathbf{y}), \\ \{\Phi_6(\mathbf{x}), \Phi_3(\mathbf{y})\} &= \frac{\partial^2 F(A, B)}{\partial A^2} \delta(\mathbf{x} - \mathbf{y}), \\ \{\Phi_6(\mathbf{x}), \Phi_4(\mathbf{y})\} &= \frac{\partial^2 F(A, B)}{\partial A \partial B} \delta(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (59)$$

For the last secondary constraint  $\Phi_7$  we obtain the non-vanishing Poisson brackets

$$\begin{aligned} \{\Phi_7(\mathbf{x}), \Phi_3(\mathbf{y})\} &= \frac{\partial^2 F(A, B)}{\partial A \partial B} \delta(\mathbf{x} - \mathbf{y}), \\ \{\Phi_7(\mathbf{x}), \Phi_4(\mathbf{y})\} &= \frac{\partial^2 F(A, B)}{\partial B^2} \delta(\mathbf{x} - \mathbf{y}), \\ \{\Phi_7(\mathbf{x}), \Phi_0\} &= \frac{1}{3} \left( \frac{2}{a} \sum_{i=1}^3 \pi^{ii} + \alpha \right). \end{aligned} \quad (60)$$

Inserting all these Poisson brackets into (56) gives the tertiary constraints

$$\dot{\Phi}_5 = N \left( -\frac{a^3B}{3} + 3\pi_\beta A \right) + \lambda_3 a^3 \approx 0. \quad (61)$$

$$\dot{\Phi}_6 = \lambda_2 + \lambda_3 \frac{\partial^2 F(A, B)}{\partial A^2} + \lambda_4 \frac{\partial^2 F(A, B)}{\partial A \partial B} \approx 0. \quad (62)$$

$$\dot{\Phi}_7 = \frac{N}{3} \left( \frac{2}{a} \sum_{i=1}^3 \pi^{ii} + \alpha \right) + \lambda_3 \frac{\partial^2 F(A, B)}{\partial A \partial B} + \lambda_4 \frac{\partial^2 F(A, B)}{\partial B^2} \approx 0. \quad (63)$$

We assume that all the second partial derivatives of  $F(A, B)$  do not vanish.<sup>1</sup> In this case the equations (61)–(63) are restrictions on the Lagrange multipliers, constituting an inhomogeneous linear equation for the unknown multipliers

---

<sup>1</sup> This is the case for example in the modified Hořava-Lifshitz gravity model  $F(\tilde{R}) \propto \tilde{R} + b\tilde{R}^2$  discussed in [5] that corresponds to

$$F(A, B) = F((3 - 9\lambda)A^2 + 2\mu B) \propto b(3 - 9\lambda)^2 A^4 + 2b\mu(3 - 9\lambda)A^2 B + 4b\mu^2 B^2 + (3 - 9\lambda)A^2 + 2\mu B,$$

so that we would have

$$\frac{\partial^2 F(A, B)}{\partial A^2} \propto 12b(3 - 9\lambda)^2 A^2 + 4b\mu(3 - 9\lambda)B + 2(3 - 9\lambda), \quad \frac{\partial^2 F(A, B)}{\partial A \partial B} \propto 4b\mu(3 - 9\lambda)A, \quad \frac{\partial^2 F(A, B)}{\partial B^2} \propto 8b\mu^2.$$

$\lambda_i, i = 2, 3, 4$ . Since the homogeneous part of this equation has only the null solution  $\lambda_2 = \lambda_3 = \lambda_4 = 0$ , the most general solution is the solution of the inhomogeneous equation:

$$\begin{aligned}\lambda_2 &= Nu_2 \equiv -\frac{N}{3} \left( B - \frac{9\pi_\beta A}{a^3} \right) \frac{\partial^2 F(A, B)}{\partial A^2} \\ &\quad + \frac{N}{3} \left[ \frac{2}{a} \sum_{i=1}^3 \pi^{ii} + \alpha + \left( B - \frac{9\pi_\beta A}{a^3} \right) \frac{\partial^2 F(A, B)}{\partial A \partial B} \right] \frac{\partial^2 F(A, B)}{\partial A \partial B} \left( \frac{\partial^2 F(A, B)}{\partial B^2} \right)^{-1}, \\ \lambda_3 &= Nu_3 \equiv \frac{N}{3} \left( B - \frac{9\pi_\beta A}{a^3} \right), \\ \lambda_4 &= Nu_4 \equiv -\frac{N}{3} \left[ \frac{2}{a} \sum_{i=1}^3 \pi^{ii} + \alpha + \left( B - \frac{9\pi_\beta A}{a^3} \right) \frac{\partial^2 F(A, B)}{\partial A \partial B} \right] \left( \frac{\partial^2 F(A, B)}{\partial B^2} \right)^{-1}.\end{aligned}\tag{64}$$

The multiplier  $\lambda_1$  is arbitrary, as is the non-dynamical variable  $N$  that also is a multiplier in the Hamiltonian (50) with (55).

The total Hamiltonian (50) can be written as a sum of two first-class constraints multiplied by the two arbitrary time-dependent multipliers  $N$  and  $\lambda_1$ :

$$H_T = NH_0 + \lambda_1 \Phi_1,\tag{65}$$

where we have defined the first-class Hamiltonian constraint by

$$H_0 = \Phi_0 + \sum_{n=2}^4 \int d^3 \mathbf{x} u_n(\mathbf{x}) \Phi_n(\mathbf{x}),\tag{66}$$

with the fields  $u_n$  ( $n = 2, 3, 4$ ) given by (64). It is easy to see that  $\Phi_1 = \pi_N$  is first-class, since it clearly has a vanishing Poisson bracket with every constraint. From (52) and (56) we see that the sum of constraints  $H_0$  is first-class by construction. Note that (66) is a combination of secondary and primary constraints. Usually a secondary first-class constraint would require us to define an extended Hamiltonian where the constraint would be added with an additional arbitrary multiplier. In this case, however, that would only lead to a redefinition of the multiplier  $N$ , and such a change ' $N \rightarrow N + \text{an arbitrary function of time}$ ' does not bring anything new to the description. As always, the first-class constraints are associated with the gauge symmetries of the system [16]. The first-class constraints  $H_0$  and  $\Phi_1$  generate the (gauge) transformations that do not change the physical state of the system.

The constraints  $\chi_k = (\Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \Phi_7)$  form the set of second-class constraints of the system.<sup>2</sup> For details on the classification and representation of second-class constraints, one can see [17]. The Poisson brackets of the second-class constraints define the matrix:

$$C_{kl}(\mathbf{x}, \mathbf{y}) \equiv \{\chi_k(\mathbf{x}), \chi_l(\mathbf{y})\} = C_{kl}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}),\tag{67}$$

where

$$C_{kl}(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -a^3 & -F_{A^2} & -F_{AB} \\ 0 & 0 & 0 & 0 & -F_{AB} & -F_{B^2} \\ 0 & a^3 & 0 & 0 & 0 & \frac{1}{3} \\ 1 & F_{A^2} & F_{AB} & 0 & 0 & 0 \\ 0 & F_{AB} & F_{B^2} & -\frac{1}{3} & 0 & 0 \end{pmatrix}\tag{68}$$

and we denote

$$F_{A^2} \equiv \frac{\partial^2 F(A, B)}{\partial A^2}, \quad F_{AB} \equiv \frac{\partial^2 F(A, B)}{\partial A \partial B}, \quad F_{B^2} \equiv \frac{\partial^2 F(A, B)}{\partial B^2}.\tag{69}$$

---

<sup>2</sup> The index of  $\chi_k$  runs over  $k = 1, 2, \dots, 6$ , so that  $\chi_k = \Phi_{k+1}$ .

This matrix has the inverse

$$C^{kl}(\mathbf{x}, \mathbf{y}) = C^{kl}(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}), \quad (70)$$

$$C^{kl}(\mathbf{x}) = \begin{pmatrix} 0 & \frac{F_{AB}}{3a^3 F_{B^2}} & -\frac{F_{A^2}}{3a^3 F_{B^2}} & \frac{F_{AB}^2 - F_{A^2} F_{B^2}}{a^3 F_{B^2}} & 1 & -\frac{F_{AB}}{F_{B^2}} \\ -\frac{F_{AB}}{3a^3 F_{B^2}} & 0 & \frac{1}{3a^3 F_{B^2}} & \frac{1}{a^3} & 0 & 0 \\ \frac{F_{A^2}}{3a^3 F_{B^2}} & -\frac{1}{3a^3 F_{B^2}} & 0 & -\frac{F_{AB}}{a^3 F_{B^2}} & 0 & \frac{1}{F_{B^2}} \\ \frac{F_{A^2} F_{B^2} - F_{AB}^2}{a^3 F_{B^2}} & -\frac{1}{a^3} & \frac{F_{AB}}{a^3 F_{B^2}} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{F_{AB}}{F_{B^2}} & 0 & -\frac{1}{F_{B^2}} & 0 & 0 & 0 \end{pmatrix}, \quad (71)$$

which satisfies

$$\int d^3 z C_{kl}(\mathbf{x}, \mathbf{z}) C^{lm}(\mathbf{z}, \mathbf{y}) = C_{kl}(\mathbf{x}) C^{lm}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) = \delta_k^m \delta(\mathbf{x} - \mathbf{y}). \quad (72)$$

Now it is possible to impose the second-class constraints  $\chi_k$  by replacing the Poisson bracket with the Dirac bracket. For any two functions or functionals  $f$  and  $h$  of the canonical variables, the Dirac bracket is defined by

$$\{f(\mathbf{x}), h(\mathbf{y})\}_{\text{DB}} = \{f(\mathbf{x}), h(\mathbf{y})\} - \int d^3 z d^3 z' \{f(\mathbf{x}), \chi_k(\mathbf{z})\} C^{kl}(\mathbf{z}, \mathbf{z}') \{\chi_l(\mathbf{z}'), h(\mathbf{y})\}. \quad (73)$$

The Dirac bracket takes fully into account how the second-class constraints impose relations between the canonical variables. Therefore it enables us to set these constraints to vanish strongly,  $\chi_k(\mathbf{x}) = 0$ . So we have the identities

$$\pi_\alpha = \pi_A = \pi_B = 0, \quad A = \frac{\pi_\beta}{3a^3} \quad (74)$$

and

$$\alpha = - \frac{\partial F(A, B)}{\partial A} \Big|_{A=\frac{\pi_\beta}{3a^3}}, \quad \beta = - \frac{\partial F(\frac{\pi_\beta}{3a^3}, B)}{\partial B}. \quad (75)$$

When the function  $F$  is known, from (75) we can solve the variable  $B$  in terms of  $\beta$  and  $\frac{\pi_\beta}{3a^3} = \frac{\pi_\beta}{3\sqrt{g}}$ :

$$B = \tilde{B}\left(\beta, \frac{\pi_\beta}{3a^3}\right). \quad (76)$$

Then  $\alpha$  can be solved:

$$\alpha = - \frac{\partial F\left(A, \tilde{B}\left(\beta, \frac{\pi_\beta}{3a^3}\right)\right)}{\partial A} \Big|_{A=\frac{\pi_\beta}{3a^3}}. \quad (77)$$

Introducing these strong constraints into the Hamiltonian gives

$$\mathcal{H} = \frac{2\pi_\beta}{3a} \sum_{i=1}^3 \pi^{ii} - a^3 \left[ \beta \tilde{B}\left(\beta, \frac{\pi_\beta}{3a^3}\right) + F\left(\frac{\pi_\beta}{3a^3}, \tilde{B}\left(\beta, \frac{\pi_\beta}{3a^3}\right)\right) \right]. \quad (78)$$

The first-class Hamiltonian (66) reduces to  $H_0 = \Phi_0$  and the total Hamiltonian becomes

$$H_T = N\Phi_0 + \lambda_1 \Phi_1 = N \int d^3 \mathbf{x} \mathcal{H} + \lambda_1 \pi_N. \quad (79)$$

The canonical variables are  $N, \pi_N, g_{ij}^{(3)}, \pi^{ij}, \beta, \pi_\beta$ . In other words  $\alpha, A, B$  and their conjugated momenta have been eliminated.

In order to obtain the equations of motion,

$$\dot{f} = \{f, H_T\}_{\text{DB}} = N\{f, \Phi_0\}_{\text{DB}} + \lambda_1 \{f, \pi_N\}_{\text{DB}}, \quad (80)$$

for the canonical variables we have to work out all the Dirac brackets (73) between the variables. We find that the Dirac bracket (73) reduces to the Poisson bracket (43) for all the canonical variables  $N, \pi_N, g_{ij}^{(3)}, \pi^{ij}, \beta, \pi_\beta$ , and consequently for any functions of these variables. In the first pair of variables,  $N$  is quite arbitrary and  $\pi_N$  does not evolve due to the equations of motion:

$$\begin{aligned}\dot{N} &= \{N, H_T\}_{\text{DB}} = \lambda_1 \{N, \pi_N\} = \lambda_1, \\ \dot{\pi}_N &= \{\pi_N, H_T\}_{\text{DB}} = \{\pi_N, N\} \int d^3\mathbf{x} \mathcal{H} = - \int d^3\mathbf{x} \mathcal{H} \approx 0,\end{aligned}\tag{81}$$

where as before  $\lambda_1$  is an arbitrary function of time. For the spatial metric we get

$$\dot{g}_{ij}^{(3)} = \{g_{ij}^{(3)}, H_T\}_{\text{DB}} = \frac{2N\pi_\beta}{3a} \delta_{ij},\tag{82}$$

where  $\dot{g}_{ij}^{(3)} = 2a\dot{a}\delta_{ij}$ . Solving for  $a$  gives

$$a(t)^3 = a(t_0)^3 + \int_{t_0}^t dt N \pi_\beta.\tag{83}$$

Hence we need  $\pi_\beta$  in order to get  $a(t)$ . This reveals that  $\pi_\beta$  does not depend on the spatial coordinate  $\mathbf{x}$ , because both  $a$  and  $N$  depend only on the time coordinate  $t$ . For the conjugated momenta we obtain the equations of motion

$$\dot{\pi}^{ij} = \{\pi^{ij}, H_T\}_{\text{DB}} = \delta^{ij} N \left( \frac{\pi_\beta}{3a^3} \sum_{k=1}^3 \pi^{kk} + \frac{3a}{2} \left[ \beta \tilde{B} + F(A, \tilde{B}) \right] - \frac{\pi_\beta}{2a^2} \frac{\partial F(A, \tilde{B})}{\partial A} \right)_{A=\frac{\pi_\beta}{3a^3}},\tag{84}$$

where the arguments of  $\tilde{B}$  are omitted for brevity,  $\tilde{B} \equiv \tilde{B}(\beta, A) = \tilde{B}(\beta, \frac{\pi_\beta}{3a^3})$ , as will be in the next equation. For the variable  $\beta$  we obtain the equation of motion

$$\dot{\beta} = \{\beta, H_T\}_{\text{DB}} = \frac{N}{3} \left( \frac{2}{a} \sum_{i=1}^3 \pi^{ii} - \frac{\partial F(A, \tilde{B})}{\partial A} \right)_{A=\frac{\pi_\beta}{3a^3}}.\tag{85}$$

For its conjugated momentum  $\pi_\beta$  we obtain the equation of motion

$$\dot{\pi}_\beta = \{\pi_\beta, H_T\}_{\text{DB}} = Na^3 \tilde{B} \left( \beta, \frac{\pi_\beta}{3a^3} \right).\tag{86}$$

Further progress in the study of dynamics practically requires one to specify the form of the function  $F$ , and then solve (76) from (75). We can conclude that when the second partial derivatives (69) of the function  $F$  are non-zero, the proposed general action (11) defines a consistent constrained theory.

Let us then briefly consider the cases when some of the second partial derivatives (69) of the function  $F$  are zero. In such cases the tertiary constraints (61)–(63) are no longer mere restrictions on the Lagrange multipliers, but in addition impose constraints on the canonical variables. As an example we consider the case when  $F_{B^2} = 0$  and  $F_{A^2} \neq 0, F_{AB} \neq 0$ . Then we obtain the tertiary constraint

$$\Phi_8 \equiv \frac{1}{3} \left( \frac{2}{a} \sum_{i=1}^3 \pi^{ii} + \alpha \right) + \left( \frac{B}{3} - \frac{3\pi_\beta A}{a^3} \right) F_{AB} \approx 0,\tag{87}$$

and solve two of the Lagrange multipliers, say  $\lambda_3$  and  $\lambda_4$ :

$$\lambda_3 = N \left( \frac{B}{3} - \frac{3\pi_\beta A}{a^3} \right), \quad \lambda_4 = -\frac{1}{F_{AB}} \left[ \lambda_2 + N \left( \frac{B}{3} - \frac{3\pi_\beta A}{a^3} \right) F_{A^2} \right],\tag{88}$$

where the third multiplier  $\lambda_2$  is arbitrary. The consistency condition  $\dot{\Phi}_8 \approx 0$  of the tertiary constraint  $\Phi_8$  imposes a quartic constraint on the canonical variables, because  $\dot{\Phi}_8$  turns out to be independent of the Lagrange multiplier  $\lambda_2$  and non-vanishing due to the constraints established so far. Further constraints may follow from the consistency condition of the quartic constraint. This has to be checked explicitly after choosing the form of the function  $F$ . These

additional constraints are a serious threat to the viability and consistency of the action, since they may delete the physical degrees of freedom. In case we also have  $F_{AB} = 0$ , we would solve the Lagrange multipliers  $\lambda_2, \lambda_3$  from (61)–(63) and obtain a tertiary constraint that restricts the field  $\beta$  to be a constant,  $\dot{\beta} \approx 0$ . Thus in the latter case we should not have introduced the auxiliary fields  $\beta$  and  $B$  in the first place, since  $F$  in the action is already linear in its second argument. We do not discuss the case  $F_{A^2} = 0$ , because it appears to have very little if any practical application.

As a specific example of the above general theory, one can consider the FRW cosmology in the modified Hořava-Lifshitz  $F(R)$  gravity studied in [5] and its further generalization considered in the present work, as action (12). However, this analysis can be used to study FRW cosmology in any theory with an action of the general form (5). Moreover, the methods presented in this section can be used to analyze any action of the form (5) in a general way, without assuming any particular space-time. The proposed modified Hořava-Lifshitz  $F(R)$  gravity will be studied in the next section.

#### IV. HAMILTONIAN ANALYSIS OF THE $F(\tilde{R})$ GRAVITY

Let us then consider the Hamiltonian analysis of the proposed action (12) for the modified Hořava-Lifshitz  $F(R)$  gravity. The analysis is similar with the Hamiltonian analysis presented in ref. [5], where a special case of this theory given by the choice (14) was proposed (see also the analysis of ref. [14]). This special case with the further restriction to the parameter value  $\mu = 0$  has been proposed and analyzed in ref. [15]. In this section we generalize the analysis of ref. [5].

By introducing two auxiliary fields  $A$  and  $B$  we can write the action (12) into a form that is linear in  $\tilde{R}$ :

$$S_{F(\tilde{R})} = \int d^4x \sqrt{g^{(3)}} N \left[ B(\tilde{R} - A) + F(A) \right]. \quad (89)$$

Then we can write  $\tilde{R}$  as

$$\tilde{R} = K_{ij} \mathcal{G}^{ijkl} K_{kl} + 2\mu \nabla_\mu (n^\mu K) - \frac{2\mu}{N} g^{(3)ij} \nabla_i^{(3)} \nabla_j^{(3)} N - \mathcal{L}_R^{(3)}(g_{ij}^{(3)}) . \quad (90)$$

Introducing (90) into (89) and performing integrations by parts yields the action

$$S_{F(\tilde{R})} = \int dt d^3x \sqrt{g^{(3)}} \left\{ N \left[ B \left( K_{ij} \mathcal{G}^{ijkl} K_{kl} - \mathcal{L}_R^{(3)}(g_{ij}^{(3)}) - A \right) + F(A) \right] - 2\mu K \left( \dot{B} - N^i \partial_i B \right) - 2\mu N g^{(3)ij} \nabla_i^{(3)} \nabla_j^{(3)} B \right\}, \quad (91)$$

where the integral is taken over the union  $\mathcal{U}$  of the  $t = \text{constant}$  hypersurfaces  $\Sigma_t$  with  $t$  over some interval in  $\mathbb{R}$ . We assume that the boundary integrals on  $\partial\mathcal{U}$  and  $\partial\Sigma_t$  vanish. The difference compared to the action studied in ref. [5] is that the potential part  $\mathcal{L}_R^{(3)}(g_{ij}^{(3)})$  may have any form that satisfies the correct scaling property under (18). In other words it is not necessarily defined by (14) and the detailed balance condition (16). However, due to the projectability condition  $N = N(t)$  the specific form of the  $\mathcal{L}_R^{(3)}(g_{ij}^{(3)})$  has very little effect on our analysis. Indeed the analysis of ref. [5] is translated to the present more general case by making the replacement (14) from rhs to lhs. Therefore we only present the main points of the generalized analysis.

In the Hamiltonian formalism the field variables  $g_{ij}$ ,  $N$ ,  $N_i$ ,  $A$  and  $B$  have the canonically conjugated momenta  $\pi^{ij}$ ,  $\pi_N$ ,  $\pi^i$ ,  $\pi_A$  and  $\pi_B$ , respectively. For the spatial metric and the field  $B$  we have the momenta

$$\pi^{ij} = \frac{\delta S_{F(\tilde{R})}}{\delta \dot{g}_{ij}} = \sqrt{g^{(3)}} \left[ B \mathcal{G}^{ijkl} K_{kl} - \frac{\mu}{N} g^{(3)ij} \left( \dot{B} - N^i \partial_i B \right) \right], \quad (92)$$

$$\pi_B = \frac{\delta S_{F(\tilde{R})}}{\delta \dot{B}} = -2\mu \sqrt{g^{(3)}} K. \quad (93)$$

We assume  $\mu \neq 0$  so that the momentum (93) does not vanish. Because the action does not depend on the time derivative of  $N$ ,  $N^i$  or  $A$ , the rest of the momenta form the set of primary constraints:

$$\pi_N \approx 0, \quad \pi^i(\mathbf{x}) \approx 0, \quad \pi_A(\mathbf{x}) \approx 0. \quad (94)$$

Due to the projectability condition also the momentum is constant on  $\Sigma_t$  for each  $t$ ,  $\pi_N = \pi_N(t)$ . Then the Hamiltonian is calculated

$$H = \int d^3\mathbf{x} (N\mathcal{H}_0 + N_i\mathcal{H}^i), \quad (95)$$

where the so-called Hamiltonian constraint and the momentum constraints are

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{\sqrt{g^{(3)}}} \left[ \frac{1}{B} \left( g_{ik}^{(3)} g_{jl}^{(3)} \pi^{ij} \pi^{kl} - \frac{1}{3} \left( g_{ij}^{(3)} \pi^{ij} \right)^2 \right) - \frac{1}{3\mu} g_{ij}^{(3)} \pi^{ij} \pi_B - \frac{1-3\lambda}{12\mu^2} B \pi_B^2 \right] \\ &\quad + \sqrt{g^{(3)}} \left[ B \left( \mathcal{L}_R^{(3)} \left( g_{ij}^{(3)} \right) + A \right) - F(A) + 2\mu g^{(3)ij} \nabla_i^{(3)} \nabla_j^{(3)} B \right], \\ \mathcal{H}^i &= -2\nabla_j^{(3)} \pi^{ij} + g^{(3)ij} \partial_j B \pi_B \\ &= -2\partial_j \pi^{ij} - g^{(3)ij} \left( 2\partial_k g_{jl}^{(3)} - \partial_j g_{kl}^{(3)} \right) \pi^{kl} + g^{(3)ij} \partial_j B \pi_B, \end{aligned} \quad (96)$$

respectively. We define the total Hamiltonian by

$$H_T = H + \lambda_N \pi_N + \int d^3\mathbf{x} (\lambda_i \pi^i + \lambda_A \pi_A), \quad (97)$$

where the primary constraints (94) are multiplied by the Lagrange multipliers  $\lambda_N$ ,  $\lambda_i$ ,  $\lambda_A$ .

The primary constraints (94) have to be preserved under time evolution of the system. Therefore we impose the secondary constraints:

$$\Phi_0 \equiv \int d^3\mathbf{x} \mathcal{H}_0 \approx 0, \quad \Phi_S^i(\mathbf{x}) \equiv \mathcal{H}^i(\mathbf{x}) \approx 0, \quad \Phi_A(\mathbf{x}) \equiv B(\mathbf{x}) - F'(A(\mathbf{x})) \approx 0. \quad (98)$$

Here the Hamiltonian constraint  $\Phi_0$  is global and the other two, the momentum constraint  $\Phi_S^i(\mathbf{x})$  and the constraint  $\Phi_A(\mathbf{x})$ , are local. It is convenient to introduce a globalised version of the momentum constraints:

$$\Phi_S(\xi_i) \equiv \int d^3\mathbf{x} \xi_i \mathcal{H}^i \approx 0, \quad (99)$$

where  $\xi_i, i = 1, 2, 3$  are arbitrary smearing functions. It can be shown that the momentum constraints  $\Phi_S(\xi_i)$  generate the spatial diffeomorphisms for the canonical variables  $B, \pi_B, g_{ij}^{(3)}, \pi^{ij}$ , and consequently for any function or functional constructed from these variables, and treats the variables  $A, \pi_A$  as constants.

The consistency of the system requires that also the secondary constraints  $\Phi_0$ ,  $\Phi_S(\xi_i)$  and  $\Phi_A(\mathbf{x})$  have to be preserved under time evolution defined by the total Hamiltonian (97), which can be written in terms of the constraints as

$$H_T = N\Phi_0 + \Phi_S(N_i) + \lambda_N \pi_N + \int d^3\mathbf{x} (\lambda_i \pi^i + \lambda_A \pi_A). \quad (100)$$

The Poisson brackets for the constraints  $\Phi_0$  and  $\Phi_S(\xi_i)$  are

$$\{\Phi_0, \Phi_0\} = 0, \quad \{\Phi_S(\xi_i), \Phi_0\} = 0, \quad \{\Phi_S(\xi_i), \Phi_S(\eta_i)\} = \Phi_S(\xi^j \partial_j \eta_i - \eta^j \partial_j \xi_i) \approx 0. \quad (101)$$

For the constraints  $\pi_A$  and  $\Phi_A(\mathbf{x})$  the Poisson brackets that do not vanish strongly are:

$$\begin{aligned} \{\pi_A(\mathbf{x}), \Phi_0\} &= -\sqrt{g^{(3)}} \Phi_A(\mathbf{x}) \approx 0, \quad \{\pi_A(\mathbf{x}), \Phi_A(\mathbf{y})\} = F''(A(\mathbf{x})) \delta(\mathbf{x} - \mathbf{y}) \\ \{\Phi_0, \Phi_A(\mathbf{x})\} &= \frac{1}{3\mu\sqrt{g^{(3)}}} \left( g_{ij}^{(3)} \pi^{ij} + \frac{1-3\lambda}{2\mu} B \pi_B \right), \quad \{\Phi_S(\xi_i), \Phi_A(\mathbf{x})\} = -\xi^i \partial_i B. \end{aligned} \quad (102)$$

Since  $F''(A) = 0$  would essentially reproduce the original projectable Hořava-Lifshitz gravity, we assume that  $F''(A) \neq 0$ . The constraint  $\Phi_A(\mathbf{x})$  can be made consistent by fixing the Lagrange multiplier  $\lambda_A$ :

$$\lambda_A = \frac{1}{F''(A)} \left( N^i \partial_i B - \frac{N}{3\mu\sqrt{g^{(3)}}} \left( g_{ij}^{(3)} \pi^{ij} + \frac{1-3\lambda}{2\mu} B \pi_B \right) \right). \quad (103)$$

Now all the constraints of the system are consistent under dynamics.

According to the Poisson brackets (101)–(102) between the constraints, we can set the second-class constraints  $\pi_A(\mathbf{x})$  and  $\Phi_A(\mathbf{x})$  to vanish strongly, and as a result turn the Hamiltonian constraint  $\Phi_0$  and the momentum constraint  $\Phi_S(\xi_i)$  into first-class constraints, by replacing the Poisson bracket with the Dirac bracket. It turns out that the Dirac bracket reduces to the Poisson bracket for any functions of the canonical variables. Assuming we can solve the constraint  $\Phi_A(\mathbf{x}) = 0$ , i.e.  $B = F'(A)$ , for  $A = \tilde{A}(B)$ , where  $\tilde{A}$  is the inverse of the function  $F'$ , we can eliminate the variables  $A$  and  $\pi_A$ . Thus the final variables of the system are  $g_{ij}^{(3)}, \pi^{ij}, B, \pi_B$ . The lapse  $N$  and the shift vector  $N_i$ , together with  $\lambda_N$  and  $\lambda_i$ , are non-dynamical multipliers. Finally the total Hamiltonian is the sum of the first-class constraints

$$H_T = N\Phi_0 + \Phi_S(N_i) + \lambda_N\pi_N + \int d^3\mathbf{x} \lambda_i \pi^i. \quad (104)$$

We conclude that also the proposed action (12) of the more general modified Hořava-Lifshitz  $F(R)$  gravity defines a consistent constrained theory when the projectability condition is postulated. For additional details and discussion on the analysis see ref. [5].

## V. HAMILTONIAN ANALYSIS OF THE $F(\tilde{R})$ GRAVITY IN FIXED GAUGE

Let us then analyze the action (12) when the gauge is fixed by (23), and we obtain the action (29). First we find the momenta canonically conjugated to the generalized coordinates  $\bar{g}_{ij}^{(3)}$  and  $\varphi$  of the action (29). For the fields  $\varphi$  and  $\bar{g}_{ij}^{(3)}$  we find the momenta

$$\pi_\varphi = \frac{\delta S_{F(\tilde{R})}}{\delta \dot{\varphi}} = \frac{\sqrt{\bar{g}^{(3)}}}{4\kappa^2} \left( -(1 - 3\lambda + 3\mu) \bar{g}^{(3)ij} \dot{\bar{g}}_{ij}^{(3)} + 3(1 - 3\lambda + 6\mu) \dot{\varphi} \right) \quad (105)$$

and

$$\bar{\pi}^{ij} = \frac{\delta S_{F(\tilde{R})}}{\delta \dot{\bar{g}}_{ij}^{(3)}} = \frac{\sqrt{\bar{g}^{(3)}}}{4\kappa^2} \left( \bar{g}^{(3)ik} \bar{g}^{(3)jl} \dot{\bar{g}}_{kl}^{(3)} - \lambda \bar{g}^{(3)ij} \bar{g}^{(3)kl} \dot{\bar{g}}_{kl}^{(3)} - (1 - 3\lambda + 3\mu) \bar{g}^{(3)ij} \dot{\varphi} \right), \quad (106)$$

respectively.

In the following analysis we will first assume

$$1 - 3\lambda + 3\mu \neq 0, \quad 1 - 3\lambda + 6\mu \neq 0, \quad \mu \neq 0, \quad (107)$$

so that the kinetic term for  $\varphi$  does not vanish. Later we will consider the special cases where these conditions do not hold. First we solve  $\dot{\varphi}$  from (105),

$$\dot{\varphi} = \frac{\frac{4\kappa^2}{\sqrt{\bar{g}^{(3)}}} \pi_\varphi + (1 - 3\lambda + 3\mu) \bar{g}^{(3)ij} \dot{\bar{g}}_{ij}^{(3)}}{3(1 - 3\lambda + 6\mu)}, \quad (108)$$

and introduce it into (106)

$$\bar{\pi}^{ij} = \frac{\sqrt{\bar{g}^{(3)}}}{4\kappa^2} \left[ \bar{g}^{(3)ik} \bar{g}^{(3)jl} \dot{\bar{g}}_{kl}^{(3)} - \left( \frac{1}{3} + \frac{3\mu^2}{1 - 3\lambda + 6\mu} \right) \bar{g}^{(3)ij} \bar{g}^{(3)kl} \dot{\bar{g}}_{kl}^{(3)} \right] - \frac{1 - 3\lambda + 3\mu}{3(1 - 3\lambda + 6\mu)} \bar{g}^{(3)ij} \pi_\varphi. \quad (109)$$

Then we find the velocities in terms of the coordinates and momenta. First we contract (109) by  $\bar{g}_{ij}^{(3)}$  and solve for

$$\bar{g}^{(3)ij} \dot{\bar{g}}_{ij}^{(3)} = -\frac{4\kappa^2}{\sqrt{\bar{g}^{(3)}}} \left( \frac{1 - 3\lambda + 6\mu}{9\mu^2} \bar{g}_{ij}^{(3)} \bar{\pi}^{ij} + \frac{1 - 3\lambda + 3\mu}{9\mu^2} \pi_\varphi \right). \quad (110)$$

This is inserted back into (109) as well as into (108), which enables us to obtain the velocities in terms of the canonical variables:

$$\dot{\varphi} = \frac{4\kappa^2}{\sqrt{\bar{g}^{(3)}}} \left( \frac{3\lambda - 1}{27\mu^2} \pi_\varphi - \frac{1 - 3\lambda + 3\mu}{27\mu^2} \bar{g}_{ij}^{(3)} \bar{\pi}^{ij} \right), \quad (111)$$

$$\dot{\bar{g}}_{ij}^{(3)} = \frac{4\kappa^2}{\sqrt{\bar{g}^{(3)}}} \left\{ \bar{g}_{ik}^{(3)} \bar{g}_{jl}^{(3)} \bar{\pi}^{kl} - \bar{g}_{ij}^{(3)} \left[ \left( \frac{1 - 3\lambda + 6\mu}{27\mu^2} + \frac{1}{3} \right) \bar{g}_{kl}^{(3)} \bar{\pi}^{kl} + \frac{1 - 3\lambda + 3\mu}{27\mu^2} \pi_\varphi \right] \right\}. \quad (112)$$

Thus there are no primary constraints. It is expected that there are no first-class constraints, since the gauge has been completely fixed by setting  $N = 1, N^i = 0$ . Due to the non-vanishing kinetic terms of  $\varphi$ , there are no second-class constraints either. The Hamiltonian is defined by

$$H = \int d^3\mathbf{x} \left( \bar{\pi}^{ij} \dot{\bar{g}}_{ij}^{(3)} + \pi_\varphi \dot{\varphi} \right) - L. \quad (113)$$

After a lengthy algebra exercise we find

$$H = \int d^3\mathbf{x} \left\{ \frac{4\kappa^2}{\sqrt{\bar{g}^{(3)}}} \left[ \frac{1}{2} \bar{g}_{ik}^{(3)} \bar{g}_{jl}^{(3)} \bar{\pi}^{ij} \bar{\pi}^{kl} - \left( \frac{1-3\lambda+6\mu}{54\mu^2} + \frac{1}{6} \right) \left( \bar{g}_{ij}^{(3)} \bar{\pi}^{ij} \right)^2 \right. \right. \\ \left. \left. - \frac{1-3\lambda+3\mu}{27\mu^2} \bar{g}_{ij}^{(3)} \bar{\pi}^{ij} \pi_\varphi + \frac{3\lambda-1}{54\mu^2} \pi_\varphi^2 \right] - \frac{\sqrt{\bar{g}^{(3)}}}{2\kappa^2} \left[ \bar{\mathcal{L}}_R^{(3)} \left( \bar{g}_{ij}^{(3)}, \varphi \right) - V(\varphi) \right] \right\}. \quad (114)$$

In fact we find that the Hamiltonian (114) is correct for any parameters  $\lambda$  and  $\mu$  as long as it is defined, i.e. when  $\mu \neq 0$ . This can be seen by considering the two cases when the kinetic cross-term vanishes ( $\dot{\varphi}$  and  $\dot{\bar{g}}_{ij}^{(3)}$  decouple),  $1-3\lambda+3\mu=0, \mu \neq 0$ , and when the kinetic  $\dot{\varphi}^2$  term vanishes,  $1-3\lambda+6\mu=0, \mu \neq 0$ , separately. Even the formulas (111)–(112) hold in these cases, though the details of their calculation are quite different. Note that the vanishing of the both kinetic terms of  $\varphi$  implies  $\mu=0$  and  $\lambda=1/3$ .

The Poisson bracket is postulated by (equal time  $t$  is understood)

$$\{\bar{g}_{ij}^{(3)}(\mathbf{x}), \bar{\pi}^{kl}(\mathbf{y})\} = \frac{1}{2} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(\mathbf{x} - \mathbf{y}), \quad \{\varphi(\mathbf{x}), \pi_\varphi(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \quad (115)$$

with all the other Poisson brackets vanishing. Now we can work out the Hamiltonian equations of motion. Because there are no constraints, the Hamiltonian (114) defines the dynamics of any function or functional  $f$  of the variables  $\bar{g}_{ij}^{(3)}, \bar{\pi}^{ij}, \varphi, \pi_\varphi$  by:

$$\dot{f} = \{f, H\}. \quad (116)$$

For the generalized coordinates  $\bar{g}_{ij}^{(3)}$  and  $\varphi$  we obtain the equations of motion (112) and (111) respectively. For the momenta  $\bar{\pi}^{ij}$  we obtain

$$\dot{\bar{\pi}}^{ij} = \frac{4\kappa^2}{\sqrt{\bar{g}^{(3)}}} \left[ \bar{g}^{(3)ij} \left( \frac{1}{4} \bar{g}_{km}^{(3)} \bar{g}_{ln}^{(3)} \bar{\pi}^{kl} \bar{\pi}^{mn} - \frac{a}{4} \left( \bar{g}_{kl}^{(3)} \bar{\pi}^{kl} \right)^2 - \frac{b}{2} \bar{g}_{kl}^{(3)} \bar{\pi}^{kl} \pi_\varphi + \frac{c}{4} \pi_\varphi^2 \right) \right. \\ \left. - \bar{g}_{kl}^{(3)} \bar{\pi}^{ik} \bar{\pi}^{jl} + a \bar{\pi}^{ij} \bar{g}_{kl}^{(3)} \bar{\pi}^{kl} + b \bar{\pi}^{ij} \pi_\varphi \right] \\ + \frac{\sqrt{\bar{g}^{(3)}}}{2\kappa^2} \left( \frac{1}{2} \bar{g}^{(3)ij} \left[ \bar{\mathcal{L}}_R^{(3)} \left( \bar{g}_{ij}^{(3)}, \varphi \right) - V(\varphi) \right] + \frac{\partial \bar{\mathcal{L}}_R^{(3)} \left( \bar{g}_{kl}^{(3)}, \varphi \right)}{\partial \bar{g}_{ij}^{(3)}} \right), \quad (117)$$

where we have introduced the constants:

$$a = \frac{1-3\lambda+6\mu}{27\mu^2} + \frac{1}{3}, \quad b = \frac{1-3\lambda+3\mu}{27\mu^2}, \quad c = \frac{3\lambda-1}{27\mu^2}. \quad (118)$$

The equation of motion for  $\pi_\varphi$  is

$$\dot{\pi}_\varphi = \frac{\sqrt{\bar{g}^{(3)}}}{2\kappa^2} \left( \frac{\partial \bar{\mathcal{L}}_R^{(3)} \left( \bar{g}_{kl}^{(3)}, \varphi \right)}{\partial \varphi} - 3A(\varphi) e^{3\varphi} \right), \quad (119)$$

where the last term is obtained from the derivative of  $V(\varphi)$  from (30)

$$\frac{dV(\varphi)}{d\varphi} = A(\varphi) \frac{dA(\varphi)}{d\varphi} F''(A(\varphi)) = 3A(\varphi) F'(A(\varphi)) = 3A(\varphi) e^{3\varphi}. \quad (120)$$



Here we have also used the definition (26) of  $\varphi$  and (27) of  $A(\varphi)$  to calculate

$$1 = \frac{d\varphi}{d\varphi} = \frac{1}{3} \frac{\frac{dA(\varphi)}{d\varphi} F''(A(\varphi))}{F'(A(\varphi))} \Rightarrow \frac{dA(\varphi)}{d\varphi} F''(A(\varphi)) = 3F'(A(\varphi)). \quad (121)$$

In particular, the equations of motion (117) for the momenta  $\bar{\pi}^{ij}$  are pretty complex. However, as always, they are first order differential equations.

The cases with  $\mu = 0$  are less interesting and we only consider them briefly. When  $\lambda = 1/3$  the field  $\varphi$  is non-dynamical and hence one has the primary constraint  $\pi_\varphi \approx 0$ . The momenta conjugate to  $g_{ij}^{(3)}$  are given by

$$\bar{\pi}^{ij} = \frac{\sqrt{\bar{g}^{(3)}}}{4\kappa^2} \left( \bar{g}^{(3)ik} \bar{g}^{(3)jl} \dot{\bar{g}}_{kl}^{(3)} - \frac{1}{3} \bar{g}^{(3)ij} \bar{g}^{(3)kl} \dot{\bar{g}}_{kl}^{(3)} \right). \quad (122)$$

It has zero trace  $\bar{g}_{ij}^{(3)} \bar{\pi}^{ij} = 0$  and can be trivially solved for  $\dot{\bar{g}}_{ij}^{(3)} = \frac{4\kappa^2}{\sqrt{\bar{g}^{(3)}}} \bar{g}_{ik}^{(3)} \bar{g}_{jl}^{(3)} \bar{\pi}^{kl}$ . When  $\lambda \neq 1/3$ , one is forced to impose the constraint  $\pi_\beta = -\bar{g}_{ij}^{(3)} \bar{\pi}^{ij} \approx 0$  that again leads to (122) and makes  $\varphi$  non-dynamical.

## VI. FRW COSMOLOGY IN POWER-LIKE MODELS: COSMIC ACCELERATION AND FUTURE SINGULARITIES

We now consider the FRW cosmology of the action (12). In the spatially-flat FRW space-time (8), since the spatial curvature vanishes,  $R_{ij}^{(3)} = R^{(3)} = 0$ , there is no contribution from  $\mathcal{L}_R^{(3)}$ , as it vanishes according to (34) or (39). In other words, the choice of  $\mathcal{L}_R^{(3)}$  in (34) or (39) gives the same FRW cosmology. Of course, this situation changes when one considers black holes or other solutions with non-trivial dependence on the spatial coordinates.

Let us first review the spatially-flat FRW equations obtained in ref. [5]. Varying the action (12) with respect to  $g_{ij}^{(3)}$  and setting  $N = 1$  one obtains:

$$0 = F(\tilde{R}) - 2(1 - 3\lambda + 3\mu) (\dot{H} + 3H^2) F'(\tilde{R}) - 2(1 - 3\lambda) H \frac{dF'(\tilde{R})}{dt} + 2\mu \frac{d^2 F'(\tilde{R})}{dt^2} + p, \quad (123)$$

where  $F'$  denotes the derivative of  $F$  with respect to its argument. Here, the matter contribution (the pressure  $p$ ) is included. On the other hand, the variation over  $N$  gives the global constraint:

$$0 = \int d^3x \left[ F(\tilde{R}) - 6 \left\{ (1 - 3\lambda + 3\mu) H^2 + \mu \dot{H} \right\} F'(\tilde{R}) + 6\mu H \frac{dF'(\tilde{R})}{dt} - \rho \right], \quad (124)$$

after setting  $N = 1$ . Here  $\rho$  is the energy density of matter and we have set again  $N = 1$ . It is important to stress that, because of the projectability condition  $N = N(t)$ , the above equation is a global constraint. If the standard conservation law is used,

$$0 = \dot{\rho} + 3H(\rho + p), \quad (125)$$

Eq. (123) can be integrated to give:<sup>3</sup>

$$0 = F(\tilde{R}) - 6 \left\{ (1 - 3\lambda + 3\mu) H^2 + \mu \dot{H} \right\} F'(\tilde{R}) + 6\mu H \frac{dF'(\tilde{R})}{dt} - \rho - \frac{C}{a^3}. \quad (126)$$

---

<sup>3</sup> Note that, as already shown in [18] for the standard case, the parameter  $\lambda$  has a crucial role in the relation of Hořava-Lifshitz type theories. In fact, from the second equation above one realizes that this solution is physical only if  $1 - 3\lambda + 3\mu > 0$ . It should also be pointed out that in Hořava-Lifshitz gravity the role of standard matter and its conservation properties are not well understood yet. We will proceed with our discussion supposing that it is possible to couple matter and gravity in the same way in which one does in GR.

Here  $C$  is the integration constant and can be set to zero. In [19], however, it has been claimed that  $C$  does not necessarily need to vanish in a local region, since (124) needs to be satisfied only in the whole universe. In this sense in a limited region, one can have  $C > 0$  and the  $Ca^{-3}$  term in (126) can be regarded as dark matter.

Note that Eq. (126) corresponds to the first FRW equation and (123) to the second one. Specifically, if we choose  $\lambda = \mu = 1$  and  $C = 0$ , Eq. (126) reduces to

$$0 = F(\tilde{R}) - 6(H^2 + \dot{H})F'(\tilde{R}) + 6H\frac{dF'(\tilde{R})}{dt} - \rho \quad (127)$$

$$= F(\tilde{R}) - 6(H^2 + \dot{H})F'(\tilde{R}) + 36(4H^2\dot{H} + \ddot{H})F''(\tilde{R}) - \rho, \quad (128)$$

which is identical to the corresponding equation in the standard  $F(R)$  gravity (see Eq. (2) in [20] where a reconstruction of the theory has been made). In the following we will explore the properties of the equations (123) and (126), especially looking for solutions that represent accelerated expansion. Solutions of this type are very important because they represent the key evolutionary phases of the universe, namely the inflationary era and the dark energy era. The connection with dark energy is particularly important to understand, if the Newtonian nature of the quantum theory of gravitation, implicit in Hořava-Lifshitz gravity, is the direct cause of cosmic acceleration and, as a consequence, of dark energy.

### A. de Sitter cosmology

Let us investigate the properties of the de Sitter solutions in this class of theories. This issue was considered for the first time in [5], but in the following a more general treatment is given. These solutions are of great importance in cosmology because they have the potential to describe both inflationary phase(s) as well as dark energy era(s). In standard  $F(R)$  gravity it has been proven that it is possible to construct a viable model unifying inflation and late time acceleration in the form of double or multiple de Sitter solution [21–24].

In vacuum ( $\rho_m, p_m = 0$ ) and substituting the de Sitter metric

$$ds^2 = -dt^2 + \exp(\gamma t) \sum_{i=1}^3 (dx^i)^2, \quad (129)$$

the equations (123) and (126) reduce to the single equation

$$F + 6\gamma^2(3\lambda - 3\mu - 1)F' = 0. \quad (130)$$

In Table I we show the values of the time constant  $\gamma$  of the de Sitter metric of some popular  $F(R)$  theories and their Hořava-Lifshitz versions.

It is interesting to note that, in contrast to what happens in standard  $F(R)$  gravity, the quadratic function  $F = \tilde{R}^m$  is not degenerate for  $m = 2$ , but only for

$$m = \frac{2}{1 - 3\lambda + 3\mu}, \quad (131)$$

i.e. it depends on the Lorentz-violation parameters.

It is also interesting to note that in general the solution of (123) and (126) is not unique. Thus a given theory can have multiple de Sitter solutions. This means that also in this case the cosmologies of these theories can admit both inflation and dark energy phases. However, since the Hořava-Lifshitz parameters are in principle only present in the coefficients of the equation (130) and the number of solutions of (130) is determined by the powers of  $\tilde{R}$  appearing in  $F$ , two corresponding theories will have in general the same number of de Sitter solutions. Obvious exceptions are the case  $3\lambda - 3\mu - 1 = 0$  for which the Eq. (130) becomes  $F = 0$  and the case  $3\lambda - 3\mu - 1 = g(\gamma)$ .

In the first case, we see that the structure of the cosmological equations is essentially changed. In particular, the equation (123) is modified and the constraint (126) loses the linear  $H^2$  term:

$$0 = F(\tilde{R}) - 6\mu\dot{H}F'(\tilde{R}) + 6\mu H\frac{dF'(\tilde{R})}{dt} - \rho \quad (132)$$

(we have considered  $C = 0$ ). Consequently, the equation (130) becomes

$$0 = F(\tilde{R}), \quad (133)$$

which is never obtained in the standard  $F(R)$  gravity.

In the second case, instead, the choice of the function  $g$  can radically change the number and type of solutions in these theories. In this sense, the solution space for Hořava-Lifshitz  $F(R)$  gravity can be considered bigger than the one of its standard counterpart. Such a fact will be even more apparent in the case of the FRW-type solutions that will be examined in the next section.

TABLE I: Some of the values of the time constant of de Sitter backgrounds for standard  $F(R)$  gravity models and their Hořava-Lifshitz counterparts. When writing the form of the function  $F$ , the Ricci scalar of both types of theories is indicated by  $x$ . For the more complex forms of  $F(R)$  an implicit equation has to be solved for the time parameter  $\gamma$  in order to find its values.

Function $F$	Standard Case	Hořava-Lifshitz case
$x + \chi x^n$	$\gamma = \pm (2^{2n-1} 3^{n-1} \alpha - 3^n 4^{n-1} n \chi)^{\frac{1}{2-2n}}$	$\gamma = \pm \left( \frac{2^{2n} 3^n \chi (3n\lambda - 3n\mu - n + 2)}{-36\lambda + 36\mu - 12} \right)^{\frac{1}{2(1-n)}}$
$x^n \exp(\chi x^m)$	$\gamma = \frac{1}{2\sqrt{3}} \left( \frac{2-n}{m\chi} \right)^{\frac{1}{2m}}$	$\gamma = \frac{1}{2\sqrt{3}} \left[ \frac{n(3\lambda - 3\mu - 1) + 2}{\chi m(3\mu + 1 - 3\lambda)} \right]^{\frac{1}{2m}}$
$\frac{x^m + \chi}{1 + \xi x^n}$	$\frac{A}{2(12^n \gamma^{2n} + \xi)^2} = 0$	$\frac{B}{2(12^n \gamma^{2n} + \xi)^2} = 0$
$x + \chi + \frac{\chi}{\alpha[(x\xi - 1)^{2n+1} + 1] + 1}$	$-C + 6\gamma^2 + \chi = 0$	$-3D + 6\gamma^2(3\lambda - 3\mu + 1) + \chi = 0$
$A = 2^{2(m+n)} 3^{m+n} (-m + n + 2) \gamma^{2(m+n)} + 2^{2m} 3^m (2 - m) \xi \gamma^{2m} + 2^{2n} 3^n (n + 2) \chi \gamma^{2n} + 2\xi \chi$ $B = 2^{2(m+n)} 3^{m+n} \gamma^{2(m+n)} (m(3\lambda - 3\mu - 1) - 3n\lambda + 3n\mu + n + 2) +$ $+ 2^{2m} 3^m \xi \gamma^{2m} (m(3\lambda - 3\mu - 1) + 2) + 2^{2n} 3^n \chi \gamma^{2n} (-3n\lambda + 3n\mu + n + 2) + 2\xi \chi$ $C = \frac{\chi(-6(-2n-3)\gamma^2\xi-1)-6(2n+1)(\alpha+1)\gamma^2\xi\chi}{(12\gamma^2\xi-1)(\alpha(12\gamma^2\xi-1)^{2n+1}+\alpha+1)^2}$ $D = \frac{6(2n+1)(\alpha+1)\gamma^2\xi\chi(3\lambda-3\mu-1)+\chi(-6\gamma^2\xi(n(6\lambda-6\mu-2)+3(\lambda-\mu-1))-1)}{(12\gamma^2\xi-1)(\alpha(12\gamma^2\xi-1)^{2n+1}+\alpha+1)^2}$		

## B. Power law solutions and reconstruction technique

In addition to the de Sitter solution described above one can also look for accelerated expansion phases in the form of power law solutions. These solution can have a double value as Friedmannian cosmologies, if the exponent of the power law is in the interval  $]0, 1[$ , and they can realize the so-called “power law inflation”, or a “power law dark energy”, if the exponent is bigger than one.

If we look for the presence of Friedmann solutions of (123) and (126), we realize quickly that, as in  $F(R)$  gravity, there is little chance to find power law solutions, unless one considers a function  $F$  of trivial form. However, due to the additional parameters, the set of solutions of this type is bigger in the Hořava-Lifshitz case than in the standard one. For example, in the simple case  $F(\tilde{R}) = \tilde{R} + \chi \tilde{R}^m$  we find that, in the presence of a barotropic fluid ( $p = w\rho$ ), the spatially-flat solution

$$a = a_0 t^{2/3(1+w)}, \quad \rho = \rho_0 t^{-2}, \quad (134)$$

satisfies (123) and (126) if

$$\mu = \frac{(w^2 - 1) \gamma (3\lambda - 1)}{2w(3(w + 1)\gamma - w + 1)}, \quad \text{and} \quad \rho_0 = \frac{4(3\lambda - 1)}{3(w + 1)^2 \kappa^2}. \quad (135)$$

This corresponds to the standard Friedmann solution. It is well known that in standard  $F(R)$  theories, the case  $F(R) = R + \chi R^m$  possesses only power law solutions of the type  $a = a_0 t$  or  $a = a_0 t^{1/2}$  (see e.g. [25]).

In order to facilitate the analysis in the next sections, we consider also some solutions for this model in the case of very small and very large scalar curvature. In the first case, the theory reduces itself to GR plus a cosmological constant and its solutions are approximated by the Friedmann ones. In the case of high curvature, instead, the theory reduces to  $F(\tilde{R}) \approx \tilde{R}^n$ . Such a theory possesses three exact solutions. The first two

$$a = a_0 t^\gamma, \quad \rho = \rho_0 t^{-2}, \quad \gamma = \frac{2m}{3(1+w)}, \quad (136)$$

$$\rho_0 = \frac{\chi [3\mu(m-1)(2m(w+2)-w-1) - m(2m-1)(3\lambda-1)]}{\kappa^2 (m[3\lambda-6\mu-1] + 3(w+1)\mu)} \left( \frac{4m^2(-3\lambda+6\mu+1) - 12m(w+1)\mu}{(w+1)^2} \right)^m,$$

and

$$a = a_0 t^\gamma, \quad \rho = \rho_0 t^{-2}, \quad \gamma = \frac{2(m-1)(2m-1)\mu}{(2m-1)(3\lambda-1) - 6(m-1)\mu}, \quad \rho_0 = 0, \quad (137)$$

correspond to the solutions in the standard  $F(R)$  case. A third solution, that is valid only for  $m > 1$ , is

$$a = a_0 t^\gamma, \quad \rho = \rho_0 t^{-2}, \quad \gamma = \frac{2\mu}{-3\lambda+6\mu+1}, \quad \rho_0 = 0, \quad (138)$$

which is characteristic of Hořava-Lifshitz gravity and does not depend on  $m$ . In the analysis of the singularities of this simple model, we will refer to these solutions.

Note that as it often happens [26] in theories of this type, the value of  $\rho_0$  can be negative (or even undefined) for certain combinations of variables. This implies that matter is not always compatible with  $F(R)$  gravity, not even in the Hořava-Lifshitz case. In our specific example  $\rho_0 > 0$  implies

$$m < 0, \quad 0 \leq w \leq 1, \quad \begin{cases} \chi < 0 & \mu > 0 & \frac{6m\mu+m-3w\mu-3\mu}{3m} < \lambda < \frac{6m^2w\mu+12m^2\mu+2m^2-9mw\mu-15m\mu-m+3w\mu+3\mu}{6m^2-3m} \\ \chi > 0 & \mu < 0 & \lambda > \frac{6m\mu+m-3w\mu-3\mu}{3m} \\ \chi > 0 & \mu \geq 0 & \lambda > \frac{6m^2w\mu+12m^2\mu+2m^2-9mw\mu-15m\mu-m+3w\mu+3\mu}{6m^2-3m} \end{cases} \quad (139)$$

$$\chi > 0, \quad \mu > 0, \quad \begin{cases} w = 0 & 0 < m < \frac{1}{2} & \lambda < \frac{6m\mu+m-3\mu}{3m} \\ w = 0 & m > \frac{1}{2} & \frac{12m^2\mu+2m^2-15m\mu-m+3\mu}{6m^2-3m} < \lambda < \frac{6m\mu+m-3\mu}{3m} \\ 0 < w \leq 1 & 0 < m < \frac{1}{2} & \lambda < \frac{6m\mu+m-3w\mu-3\mu}{3m} \\ 0 < w \leq 1 & \frac{1}{2} < m < \frac{1+w}{2w} & \frac{6m^2w\mu+12m^2\mu+2m^2-9mw\mu-15m\mu-m+3w\mu+3\mu}{6m^2-3m} < \lambda < \frac{6m\mu+m-3w\mu-3\mu}{3m} \end{cases} \quad (140)$$

$$\chi < 0, \quad \mu > 0, \quad \begin{cases} w = 0 & m > \frac{1}{2} & \lambda < \frac{12m^2\mu+2m^2-15m\mu-m+3\mu}{6m^2-3m} \\ 0 < w \leq 1 & \frac{1}{2} < m < \frac{1+w}{2w} & \lambda < \frac{6m^2w\mu+12m^2\mu+2m^2-9mw\mu-15m\mu-m+3w\mu+3\mu}{6m^2-3m} \\ 0 < w \leq 1 & m > \frac{1+w}{2w} & \lambda < \frac{6m\mu+m-3w\mu-3\mu}{3m} \end{cases} \quad (141)$$

$$\chi > 0, \quad \mu < 0, \quad \begin{cases} w = 0 & m > \frac{1}{2} & \lambda < \frac{12m^2\mu+2m^2-15m\mu-m+3\mu}{6m^2-3m} \\ 0 < w \leq 1 & \frac{1}{2} < m < \frac{1+w}{2w} & \lambda < \frac{6m^2w\mu+12m^2\mu+2m^2-9mw\mu-15m\mu-m+3w\mu+3\mu}{6m^2-3m} \\ 0 < w \leq 1 & m > \frac{1+w}{2w} & \frac{6m^2w\mu+12m^2\mu+2m^2-9mw\mu-15m\mu-m+3w\mu+3\mu}{6m^2-3m} < \lambda < \frac{6m\mu+m-3w\mu-3\mu}{3m} \end{cases} \quad (142)$$

$$\chi < 0, \quad \mu < 0, \quad \begin{cases} w = 0 & 0 < m < \frac{1}{2} & \frac{12m^2\mu+2m^2-15m\mu-m+3\mu}{6m^2-3m} < \lambda < \frac{6m\mu+m-3\mu}{3m} \\ w = 0 & m > \frac{1}{2} & \lambda < \frac{6m\mu+m-3\mu}{3m} \\ 0 < w \leq 1 & 0 < m < \frac{1}{2} & \frac{6m^2w\mu+12m^2\mu+2m^2-9mw\mu-15m\mu-m+3w\mu+3\mu}{6m^2-3m} < \lambda < \frac{6m\mu+m-3w\mu-3\mu}{3m} \\ 0 < w \leq 1 & \frac{1}{2} < m < \frac{1+w}{2w} & \lambda < \frac{6m\mu+m-3w\mu-3\mu}{3m} \end{cases} \quad (143)$$

Another result which will be useful for our purposes is an exact solution for the theory  $F(\tilde{R}) = \tilde{R} + \xi \tilde{R}^2 + \chi \tilde{R}^m$ . For  $m > 2$ , this solution reads

$$\begin{aligned} a &= a_0 t^\gamma, & \gamma &= \frac{2}{3(1+w)}, & \mu &= \frac{1-3\lambda}{3(w-1)}, \\ \rho &= \rho_0 t^{-2}, & \rho_0 &= \frac{4(3\lambda-1)}{3(w+1)^2 \kappa^2}. \end{aligned} \quad (144)$$

This solution is obviously present only in the Hořava-Lifshitz version of this theory, as one can check directly.

One of the most important methods used to investigate power law solutions in higher order gravity is the reconstruction of a theory starting from a specific background. In the following we will adapt this technique to reconstruct the form of the function  $F(\tilde{R})$  that admits flat FRW power law solutions [20, 27–31].

Let us then consider a cosmological solution characterized by the Hubble parameter

$$H = \frac{\gamma}{t}, \quad (145)$$

and again assuming the energy density of a barotropic fluid

$$\rho = \rho_0 t^{-3\gamma(1+w)}. \quad (146)$$

In this case the Ricci scalar is

$$\tilde{R} = \frac{3\gamma(-3\gamma\lambda + 6\gamma\mu + \gamma - 2\mu)}{t^2}, \quad (147)$$

so that we can express the time  $t$  as a function of  $\tilde{R}$ . Substituting (145) and (146) into (123) and (126) and expressing  $t$  in terms of  $\tilde{R}$ , one obtains

$$A_1 \tilde{R}^3 F^{(3)} + A_2 \tilde{R}^2 F'' + A_3 \tilde{R} F' + A_4 F + ew \tilde{R}^{\frac{3}{2}(w+1)\gamma} = 0, \quad (148)$$

$$B_1 \tilde{R}^2 F'' + B_2 \tilde{R} F' + B_3 F + B_4 \tilde{R}^{\frac{3}{2}(w+1)\gamma} = 0, \quad (149)$$

with

$$A_1 = \frac{8\mu}{3\gamma(\gamma - 3\gamma\lambda + 6\gamma\mu - 2\mu)}, \quad (151)$$

$$A_2 = -\frac{4(\gamma - 3\gamma\lambda + 3\mu)}{3\gamma(\gamma(3\lambda - 6\mu - 1) + 2\mu)}, \quad (152)$$

$$A_3 = \frac{2(3\gamma - 1)(3\lambda - 3\mu - 1)}{3(\gamma - 3\gamma\lambda + 6\gamma\mu - 2\mu)}, \quad (153)$$

$$A_4 = 3^{-\frac{3}{2}(w+1)\gamma} \kappa^2 w \rho_0 [\gamma^2(1 - 3\lambda + 6\mu) - 2\mu\gamma]^{-\frac{3}{2}(w+1)\gamma} \quad (154)$$

and

$$B_1 = \frac{4\mu}{3\gamma\lambda - 6\gamma\mu - \gamma + 2\mu}, \quad (155)$$

$$B_2 = \frac{\gamma(3\lambda - 1)}{\gamma - 3\gamma\lambda + 6\gamma\mu - 2\mu} - 1, \quad (156)$$

$$B_3 = 1, \quad (157)$$

$$B_4 = \kappa^2 \rho_0 \left( -3^{-\frac{3}{2}(w+1)\gamma} \right) (\gamma(\gamma - 3\gamma\lambda + 6\gamma\mu - 2\mu))^{-\frac{3}{2}(w+1)\gamma}. \quad (158)$$

These equations admit the solution

$$F(\tilde{R}) = C_1 \tilde{R}^{\alpha_-} + C_2 \tilde{R}^{\alpha_+} + C_3 \tilde{R}^{\frac{3}{2}(1+w)\gamma}, \quad (159)$$

where

$$\alpha_{\pm} = \frac{\gamma(3\lambda - 3\mu - 1) + 3\mu \pm \sqrt{\gamma^2(-3\lambda + 3\mu + 1)^2 + 2\gamma\mu(3\lambda + 3\mu - 1) + \mu^2}}{4\mu} \quad (160)$$

and

$$C_3 = \frac{3^{-\frac{3}{2}(w+1)\gamma} \kappa^2 \rho_0 [\gamma(-3\gamma\lambda + 6\gamma\mu + \gamma - 2\mu)]^{1-\frac{3}{2}(w+1)\gamma}}{\gamma[\gamma(3\lambda - 1)(3(w+1)\gamma - 1) - \mu(3(w+1)\gamma - 2)(3(w+2)\gamma - 1)]}. \quad (161)$$

Note that the coefficients  $\alpha$  are real only for

$$\gamma^2 (-3\lambda + 3\mu + 1)^2 + 2\gamma\mu (3\lambda + 3\mu - 1) + \mu^2 > 0,$$

which is satisfied for

$$\gamma \geq 0 \tag{162}$$

$$\gamma < 0, \quad \text{and} \quad \lambda < \frac{3\gamma\mu + \gamma - \mu}{3\gamma} - 2\sqrt{-\frac{\mu^2}{3\gamma}} \quad \lambda > \frac{3\gamma\mu + \gamma - \mu}{3\gamma} + 2\sqrt{-\frac{\mu^2}{3\gamma}}. \tag{163}$$

Therefore also in the Hořava-Lifshitz case, the only type of function  $F$  that is able to generate analytical power law solutions is a combination of powers of the Ricci scalar. The connection between the equations (145) and (159) allows one to make some general considerations on the relation between the structure of the function  $F$  and the cosmology. If one plots the behavior of the exponents of (159) as a function of  $\gamma$  for various values of  $\mu$  and  $\lambda$  (see figure 1), one finds that there is a correlation between the existence of a specific type of solutions and the sign of the modes of the reconstructed theory. For example, if one chooses only positive values for the exponents of (159) in order to avoid instabilities, none of the permitted values of the parameters are able to generate a contracting solution. On the other hand, both a Friedmann expansion and a power law inflation regimes can be obtained if  $\mu < 0$  and  $\lambda < \frac{6\gamma\mu + \gamma - 2\mu}{3\gamma}$  or  $\mu > 0$  and  $\lambda > \frac{6\gamma\mu + \gamma - 2\mu}{3\gamma}$ . The behavior of the exponents of (159) is shown in Figure 1 for different values of the parameters.

As a final remark, it is interesting to note that the case  $F(\tilde{R}) = C_2\tilde{R} + C_3\tilde{R}^m$  corresponds to the solution (136) via the reconstruction method. This conclusion confirms the correctness of this approach and its utility in the search for exact solutions.

### C. Explicit model for the unification of inflation with dark energy

It is interesting to try to formulate an explicit model for the unification of early-time inflation with late-time acceleration. One may propose such a model, which may unify the inflation and the late-time acceleration. First we consider the case  $3\lambda - 3\mu - 1 = 0$ , which is specific in the Hořava-Lifshitz  $F(R)$  gravity. An example is

$$F(R) = \frac{1}{2\kappa^2} (1 + c_1 \ln \kappa^2 R) (1 - c_2 \kappa^2 R). \tag{164}$$

Here  $c_1$  and  $c_2$  are dimensionless positive constants. Then following (133), we find two de Sitter solutions

$$R = R_L \equiv \kappa^2 e^{-\frac{1}{c_1}}, \quad R = R_I \equiv \frac{1}{c_2 \kappa^2}. \tag{165}$$

If one chooses  $c_1 \sim 1/280$ , we find  $R_L \sim (10^{-33} \text{ eV})^2$ , which may describe the accelerating expansion of the present universe. On the other hand, if  $c_2 \sim \mathcal{O}(1) - \mathcal{O}(100)$ ,  $R = R_I$  may express the inflation.

In the general case  $3\lambda - 3\mu - 1 \neq 0$ , we may consider the following form of  $F(\tilde{R})$ :

$$F(\tilde{R}) = \tilde{R} + f(\tilde{R}) \quad f(\tilde{R}) = R_I \tanh \frac{\tilde{R} - R_1}{\Lambda} + R_L \tanh \frac{\tilde{R} - R_2}{\Lambda} + R_I \tanh \frac{R_1}{\Lambda} + R_L \tanh \frac{R_2}{\Lambda}. \tag{166}$$

Here  $R_I$ ,  $R_L$ ,  $R_1$ ,  $R_2$ , and  $\Lambda$  are positive constants and we assume

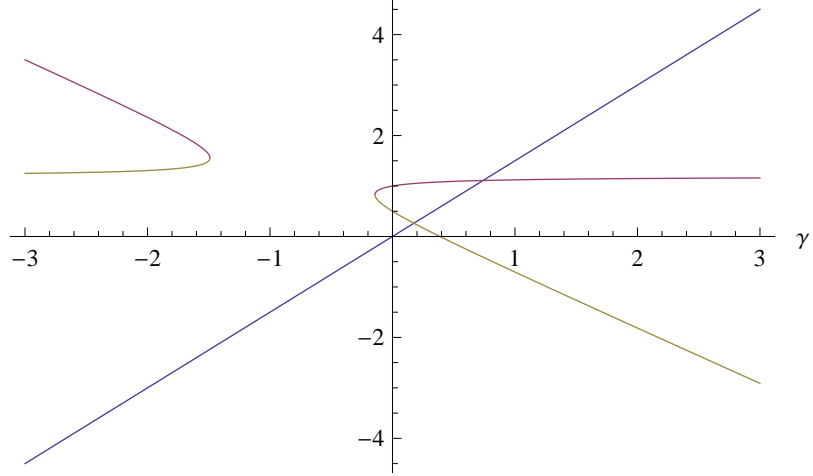
$$R_I \gg R_L \gg \Lambda \quad R_I \gg R_1 \quad R_L \gg R_2. \tag{167}$$

Then, when  $\tilde{R} \sim R_I$ , we find

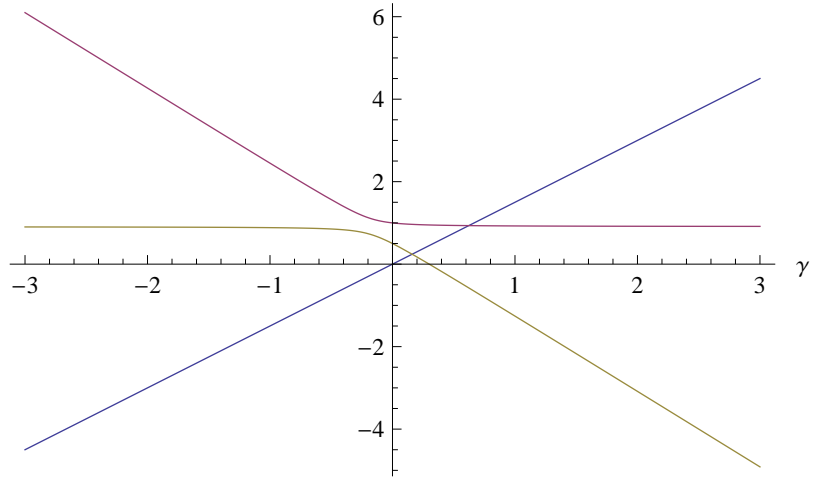
$$f(\tilde{R}) \sim R_I, \tag{168}$$

such that  $f(\tilde{R})$  plays the role of a large cosmological constant, which generates inflation. On the other hand, when  $\tilde{R} \sim R_L$ ,  $f(\tilde{R})$  becomes a small constant,

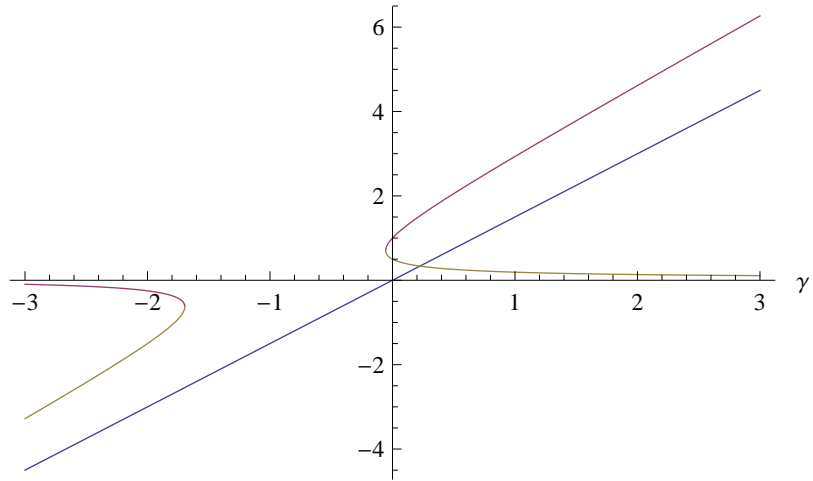
$$f(\tilde{R}) \sim R_L, \tag{169}$$



(a) The exponents of (159) for  $w = 0$ ,  $\mu = -3$  and  $\lambda = -1/2$ .



(b) The exponents of (159) for  $w = 0$ ,  $\mu = -3$  and  $\lambda = 1$ .



(c) The exponents of (159) for  $w = 0$ ,  $\mu = -2/5$  and  $\lambda = 1/2$ .

FIG. 1: Plot of the curves representing the values of the exponents of the reconstructed  $F(R)$  theory corresponding to a background  $H = \gamma/t$ , for  $w = 0$  and some specific values of the parameters  $\lambda$  and  $\mu$ . From these plots one can infer, for example, that some backgrounds of this type can only be realized in  $F(R)$  theories with poles.

and the late time acceleration could be generated. Note that

$$f(0) = 0, \quad (170)$$

therefore  $f(\tilde{R})$  is not real cosmological constant. Hence, explicit construction of realistic models for the unification of the inflation with dark energy is possible. The remaining freedom in the choice of parameters gives the possibility to make the model quite satisfactory from the cosmological point of view.

#### D. Finite-time future singularities

The attempts of constructing accelerating cosmological models that include a dark component have revealed that such models often contain some unexpected phenomenology. One of the most striking features of dark energy cosmologies is that, regardless of the way in which the dark component is introduced, they can become singular. By “become singular” we intend that there exists a specific time  $t_s$  in which one of more key quantities of the model becomes divergent. Some of these singularities, like the so-called “Big Rip” [32], are realized only far in the future (i.e.  $t_s \gg 1$ ). However, as it was recently pointed out [33, 34], under special circumstances one can have quintessence-like cosmologies that present softer singularities at smaller finite time (“sudden singularities”). Although in pure GR cosmologies this pathological behavior can be cured by specifying, for example, the equation of state of the fluid, this is not the case in models that include dark fluids or in modified gravity. In fact, it has been proved that these theories can admit up to four different types of singularities at finite time [35]. In the following, we will analyze the presence of these singularities in Hořava-Lifshitz  $F(R)$  gravity using some specific examples.

Let us define the effective density and the effective pressure associated to the Hořava-Lifshitz  $F(R)$  gravity. We have

$$\begin{aligned} \rho_{\text{eff}} &= \frac{1}{\kappa^2} \left[ 6\mu H \dot{R} F'' - 6\mu \dot{H} F' + 6(3\lambda - 3\mu - 1) H^2 F' + F \right], \\ p_{\text{eff}} &= \frac{1}{\kappa^2} \left[ -2\mu F^{(3)} \dot{R}^2 - 2(3\lambda + \mu - 1) H \dot{R} F'' - 2(3\lambda - 3\mu - 1) \dot{H} F' - 6(3\lambda - 3\mu - 1) H^2 F' - F \right]. \end{aligned} \quad (171)$$

The different types of finite-time singularities can be classified by looking at the behavior of the quantities (171) plus the scale factor  $a$ ,  $H$  and its derivatives. In particular, in [35] they are classified as

- Type I (“Big Rip”) : For  $t \rightarrow t_s$ ,  $a \rightarrow \infty$  and  $(\rho, |p|) \rightarrow \infty$  or  $\rho$  and  $|p|$  are finite;
- Type II (“sudden”) : For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$ ,  $\rho \rightarrow \rho_s$  and  $|p| \rightarrow \infty$ ;
- Type III : For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$ ,  $\rho \rightarrow \infty$  and  $|p| \rightarrow \infty$ ;
- Type IV : For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$  ( $\rho, |p|$ )  $\rightarrow$  constant (or zero) and higher derivatives of  $H$  diverge.

To classify the singularities in our case, let us consider the case of a vacuum spatially-flat cosmology, and let us imagine that close to the time  $t_s$  the Hubble parameter can be written as

$$H \approx h_0 (t_s - t)^{-\gamma}. \quad (172)$$

This means that the scale factor is

$$a \approx a_0 \exp \left[ \frac{h_0 (t_s - t)^{1-\gamma}}{\gamma - 1} \right], \quad (173)$$

The above expression tells us that there are two possible behaviors of the scale factor that depend on the value of  $\gamma$ : if  $\gamma \geq 1$ ,  $a$  will diverge as  $t$  approaches  $t_s$  while if  $\gamma < 1$ ,  $a$  will converge. Therefore it is clear that the singularity of type I is realized when  $\gamma > 1$ , the others for  $\gamma < 1$ .

The value of  $\gamma$  also influences the form of the Ricci scalar. In general, one has

$$\tilde{R} \approx h_0^2 (-9\lambda + 18\mu + 3) (t_s - t)^{-2\gamma} - 6\gamma h_0 \mu (t_s - t)^{-(\gamma+1)}, \quad (174)$$



but, depending on the value of  $\gamma$ , the above expression can be reduced to

$$\tilde{R} \approx \begin{cases} h_0^2 (-9\lambda + 18\mu + 3) (t_s - t)^{-2\gamma} & \text{if } \gamma > 1, \\ 6h_0\gamma\mu(t_s - t)^{-\gamma-1} & \text{if } \gamma < 1. \end{cases} \quad (175)$$

This property of the curvature also indicates that for  $t \rightarrow t_s$  one has

$$\begin{cases} \tilde{R} \gg 1 & \gamma > -1, \\ \tilde{R} \ll 1 & \gamma < -1, \end{cases} \quad (176)$$

i.e. the curvature may become divergent or very small depending on the value of  $\gamma$ . This property will turn out to be very useful for simplifying the calculations.

Finally, because of the nature of the structure of the effective energy density and pressure, theories with the different types of singularities are associated directly with the values of  $\gamma$ . Specifically:

- Type I  $\Rightarrow \gamma > 1$ ;
- Type II  $\Rightarrow -1 < \gamma < 0$ ;
- Type III  $\Rightarrow 0 < \gamma < 1$ ;
- Type IV  $\Rightarrow \gamma < -1$ .

Let us now consider some simple examples in the Hořava-Lifshitz  $F(R)$  gravity and their comparison with the standard case.

$$1. \quad \text{The case } F(\tilde{R}) = \tilde{R} + \chi\tilde{R}^m$$

In the case

$$F(\tilde{R}) = \tilde{R} + \chi\tilde{R}^m, \quad (177)$$

substituting (172) and (175), one finds the necessary conditions for the presence of the singularities:

$$\begin{array}{ll} \text{Type I} & \gamma > 1 \\ \text{Type II} & m < 0, \quad -1 < \gamma < 0 \\ \text{Type III} & 0 < \gamma < 1, \quad m \neq 0 \\ \text{Type IV} & \gamma \in \mathbb{Q} - \mathbb{Z}, \quad \gamma < -1. \end{array} \quad (178)$$

These are compatible with the solutions found in [34]. It is important to stress that the conditions (178) are only necessary for the existence of the singularity. The reason is that we have implicitly postulated that the solution (172) satisfies (123) and (126) at least in a specific time interval, which may not be the case due to the non-linearity of the theory. Then the only way to proceed is to find some exact solutions of the theory we are examining and see if the parameters have values for which the conditions (178) are satisfied. Unfortunately finding exact solutions in these theories can be problematic. However, close to the singularity one can solve this problem by using the fact that, as we have seen, the magnitude of the Ricci scalar also changes when  $t \rightarrow t_s$ . This means that we can approximate the function  $F$  with a simpler form that admits simple exact solutions. Then these solutions can be used to obtain necessary and sufficient conditions for the singularity to be realized.

In our simple example it is clear that one has

$$F \approx \begin{cases} \tilde{R}^n & \gamma > -1, \\ \tilde{R} & \gamma < -1. \end{cases}$$

This means that we can approximate our  $F$  with  $R^n$  in all the cases of interest and that we can use the exact solutions found in Section VIB in order to understand the presence of singularities. In particular, one sees that the solution

(137) can present a singularity of Type I for

$$\lambda < \frac{1}{3}, \quad \begin{cases} 0 < m < \frac{1}{2} & \frac{6m\lambda-2m-3\lambda+1}{8m^3-16m^2+16m-8} < \mu < \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8}, \\ \frac{1}{2} < m < 1 & \frac{6m\lambda-2m-3\lambda+1}{6m-6} < \mu < \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8}, \\ 1 < m < 2 & \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8} < \mu < \frac{6m\lambda-2m-3\lambda+1}{6m-6}, \\ m = 2 & \mu < \frac{1}{6}(9\lambda-3), \\ m > 2 & \mu < \frac{6m\lambda-2m-3\lambda+1}{6m-6} \quad \mu > \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8}, \end{cases} \quad (179)$$

$$\lambda = \frac{1}{3}, \quad m > 2 \quad \mu \neq 0 \quad (180)$$

$$\lambda > \frac{1}{3}, \quad \begin{cases} 0 < m < \frac{1}{2} & \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8} < \mu < \frac{6m\lambda-2m-3\lambda+1}{8m^3-16m^2+16m-8}, \\ \frac{1}{2} < m < 1 & \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8} < \mu < \frac{6m\lambda-2m-3\lambda+1}{6m-6}, \\ 1 < m < 2 & \frac{6m\lambda-2m-3\lambda+1}{6m-6} < \mu < \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8}, \\ m = 2 & \mu > \frac{1}{6}(9\lambda-3), \\ m > 2 & \mu < \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8} \quad \mu > \frac{6m\lambda-2m-3\lambda+1}{6m-6}, \end{cases} \quad (181)$$

a singularity of Type II for

$$\lambda < \frac{1}{3}, \quad \begin{cases} m < -1 & 0 < \mu < \frac{6m\lambda-2m-3\lambda+1}{4m^2-4}, \\ m = -1 & \mu > 0, \\ -1 < m < 0 & \mu < \frac{6m\lambda-2m-3\lambda+1}{4m^2-4} \quad \mu > 0, \end{cases} \quad (182)$$

$$\lambda = \frac{1}{3}, \quad -1 < m < 0 \quad \mu \neq 0 \quad (183)$$

$$\lambda > \frac{1}{3}, \quad \begin{cases} m < -1 & \frac{6m\lambda-2m-3\lambda+1}{4m^2-4} < \mu < 0, \\ m = -1 & \mu < 0, \\ -1 < m < 0 & \mu < 0 \quad \mu > \frac{6m\lambda-2m-3\lambda+1}{4m^2-4}, \end{cases} \quad (184)$$

and a singularity of Type III for

$$\lambda < \frac{1}{3}, \quad \begin{cases} m < 0 & \mu < \frac{6m\lambda-2m-3\lambda+1}{6m-6}, \\ 0 < m < \frac{1}{2} & \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8} < \mu < 0, \\ \frac{1}{2} < m < 1 & \mu < 0 \quad \mu > \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8}, \\ 1 < m < 2 & \mu < \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8} \quad \mu > 0, \\ m = 2 & \mu > 0, \\ m > 2 & 0 < \mu < \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8}, \end{cases} \quad (185)$$

$$\lambda = \frac{1}{3}, \quad \frac{1}{2} < m < 2 \quad m \neq 1 \quad \mu \neq 0 \quad (186)$$

$$\lambda > \frac{1}{3}, \quad \begin{cases} m < 0 & \mu < 0 \quad \mu > \frac{6m\lambda-2m-3\lambda+1}{6m-6}, \\ 0 < m < \frac{1}{2} & 0 < \mu < \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8}, \\ \frac{1}{2} < m < 1 & \mu < \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8} \quad \mu > 0, \\ 1 < m < 2 & \mu < 0 \quad \mu > \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8}, \\ m = 2 & \mu < 0 \\ m > 2 & \frac{-6m\lambda+2m+3\lambda-1}{4m^2-12m+8} < \mu < 0. \end{cases} \quad (187)$$

This is very different from the standard case, where we have a singularity of Type I only for  $m > 2$ , and a Type III for  $\frac{1}{2} < m < 1$  and  $m < 0$ .

For the solution (136) we have a singularity of Type III when

$$\begin{cases} -3 < m \leq -\frac{3}{2} & \frac{1}{3}(-2m-3) < w \leq 1, \\ -\frac{3}{2} < m < \frac{1}{2}(1-\sqrt{13}) & 0 \leq w \leq 1. \end{cases}$$

For the new solution (138), which exists only in the Hořava-Lifshitz version of  $F(R)$  gravity, we have instead a singularity of Type I for

$$\begin{cases} \lambda < \frac{1}{3} & \frac{1}{6}(3\lambda - 1) < \mu < \frac{1}{8}(3\lambda - 1) & m > 1, \\ \lambda > \frac{1}{3} & \frac{1}{8}(3\lambda - 1) < \mu < \frac{1}{6}(3\lambda - 1) & m > 1, \end{cases}$$

and a Type III for

$$\begin{cases} \lambda < \frac{1}{3} & \frac{1}{8}(3\lambda - 1) < \mu < 0 & m > 1, \\ \lambda > \frac{1}{3} & 0 < \mu < \frac{1}{8}(3\lambda - 1) & m > 1. \end{cases}$$

The results above show clearly that the presence of singularities is deeply altered in the Hořava-Lifshitz version of  $F(R)$  gravity. In particular, it seems that the additional parameters make it much easier to realize the singularities. The intervals we have presented above for the parameters can then be interpreted as constraints on this type of Hořava-Lifshitz  $F(R)$  theories of gravity. Thus, we have demonstrated that modified Hořava-Lifshitz gravity has the phantom-like or quitescence-like accelerating cosmologies, which might lead to singularities of type I, II, or III.

## 2. Eliminating the singularities

Using conformal techniques, it was first argued in [36] that in the standard  $F(R)$  gravity the singularities of the type we have found can be cured, when additional powers of the Ricci scalar are added to the Lagrangian. We can then verify if something similar happens also in the Hořava-Lifshitz case. Let us therefore consider the case

$$F(\tilde{R}) = \tilde{R} + \xi \tilde{R}^2 + \chi \tilde{R}^m. \quad (188)$$

The action of this theory contains a correction of order  $\tilde{R}^2$  and, in the standard case, it is able to cure the singularities of theories of the type  $\tilde{R} + \chi \tilde{R}^m$ .

If one derives the necessary conditions for the presence of a singularity, one finds:

$$\begin{array}{ll} \text{Type I} & \gamma > 1, \\ \text{Type II} & \text{never}, \\ \text{Type III} & 0 < \gamma < 1 \quad m \neq 0, \\ \text{Type IV} & \gamma \in \mathbb{Q} - \mathbb{Z} \quad \gamma < -1. \end{array} \quad (189)$$

One can already see at this level that in this kind of theory singularities of Type II never occur. This can be interpreted as the fact that the correction  $R^2$  is able to compensate for these terms.

As said before, the conditions above are only necessary and the only way to actually determine if a singularity is present is to consider an exact solution of the theory and see if the conditions above can be satisfied by that solution. Let us consider then the solution (144) found in the previous section. Applying the conditions above one obtains that none of them is satisfied for this background. In other words, the addition of the  $\tilde{R}^2$  term compensates for the singularities one would find in a similar background of (177). This supports the claim made in [36] that (188) is more regular than (177) and that in general the introduction of additional curvature invariants into the action can help in the curing of the singularities of an  $F(R)$  theory of gravity. On the other hand, it is also quite possible that the account of quantum gravity effects [37] may also cure the future singularities.

Summarizing, because of the additional parameters, the Hořava-Lifshitz version of  $F(R)$  gravity has a bigger space of de Sitter solutions compared to its standard counterpart. Using a simple theory and both a direct resolution of the cosmological equations and a reconstruction technique, we have also verified that this is true in the case of power law solutions. In general the presence of additional parameters also means that one has a bigger freedom in the choice of the features of these solutions, e.g. to see if it is possible to realize accelerated expansion. Therefore in these theories the realization of dark energy era (and inflation) is comparatively easier. This is interesting because it draws a direct connection between the Newtonian nature of quantum gravity and the observed behavior of the Universe. Unfortunately, the additional number of parameters also increases the probability that these cosmologies will become singular not only at  $t \rightarrow \infty$ , but also at finite time. Using exact solutions, we have shown that there are many combinations of values of the parameters of the theory which are able to induce the appearance of singularities.

We have also verified, via a specific example, that adding an invariant of the type  $R^2$  into the Lagrangian of a theory, one is able to obtain a theory whose solutions are much more regular. Consequently, also in the Hořava-Lifshitz case one can compensate such ill behavior of these models by the introduction of additional powers of the Ricci scalar into the action.

## VII. CORRECTIONS TO THE NEWTON LAW

Let us now consider the possible corrections to the Newton law. For this purpose, we consider the infrared region where the higher derivative terms like (34) or (39) can be neglected and we find

$$\mathcal{L}_R^{(3)} \left( g_{ij}^{(3)} \right) \sim R^{(3)}. \quad (190)$$

The  $R^{(3)}$  term can be added from the beginning or this term might be induced at the infrared fixed point [3]. Then by the transformation (28), by using (35) and (30), in (29), one gets

$$\begin{aligned} & \int d^4x \sqrt{\bar{g}^{(3)}} \left\{ \bar{\mathcal{L}}_R^{(3)} \left( \bar{g}_{ij}^{(3)}, \varphi \right) - V(\varphi) \right\} \\ &= \int d^4x \sqrt{\bar{g}^{(3)}} \left\{ e^\varphi \left( \bar{R}^{(3)} - \frac{5}{2} \bar{g}^{(3)kl} \bar{\nabla}_k^{(3)} \varphi \bar{\nabla}_l^{(3)} \varphi \right) - (A(\varphi) F'(A(\varphi))) - F(A(\varphi)) \right\}. \end{aligned} \quad (191)$$

The usual Newton law can be generated through the exchange of the graviton. Furthermore by the exchange of the scalar field  $\varphi$ , extra force might be generated. Now we consider the case that

$$F(A) = A - \Lambda_{\text{eff}} - \frac{c}{A^n} + \mathcal{O}(A^{-n-1}). \quad (192)$$

Here  $\Lambda_{\text{eff}}$  is an effective cosmological constant. Then

$$F'(A) = 1 + \frac{cn}{A^{n+1}} + \dots. \quad (193)$$

In the solar system or on the earth, the second term in (193) is much smaller than unity, which corresponds to the first term. Hence, we find

$$\varphi \equiv \frac{1}{3} \ln F'(A) \sim \frac{cn}{3A^{n+1}}, \quad (194)$$

and therefore

$$V(\varphi) \sim \Lambda_{\text{eff}} + \frac{c(n+1)}{A^n} \sim \Lambda_{\text{eff}} + \frac{(n+1)c^{\frac{1}{n+1}}}{n^{\frac{n}{n+1}}} \varphi^{-\frac{n}{n+1}}. \quad (195)$$

Then since  $e^\varphi \sim 1$ , the effective mass  $m_\varphi$  of  $\varphi$  is given by

$$m_\varphi^2 \equiv \frac{V''(\varphi)}{5} \sim \frac{n^{\frac{1}{n+1}}(2n+1)c^{\frac{1}{n+1}}}{n+1} \varphi^{-\frac{3n+2}{n+1}} \sim \frac{3^{\frac{3n+2}{n+1}}(2n+1)}{n^{\frac{3n+1}{n+1}}(n+1)c^{\frac{3n+1}{n+1}}} A^{3n+2}. \quad (196)$$

In the “realistic” model,  $c$  is chosen to be  $c = \mu^{2(n+1)}$  and  $\mu \sim 10^{-33} \text{ eV}$ . On the other hand, we find  $A = R \sim 10^{-61} \text{ eV}^2$  in the solar system and  $A = R \sim 10^{-50} \text{ eV}^2$  on the earth. Thus, one finds  $m_\varphi^2 \sim 10^{15n-56} \text{ eV}^2$  in the solar system and  $m_\varphi^2 \sim 10^{48n-34} \text{ eV}^2$ . Hence, if  $n$  could be large enough, the mass of  $\varphi$  would become large and the Compton length would become small, so that the correction to the Newton law would not be observed.

## VIII. DISCUSSION AND CONCLUSIONS

We have proposed a first-order modified Hořava-Lifshitz-like gravity action and studied its Hamiltonian structure. As a large explicit class of such models we considered the modified Hořava-Lifshitz  $F(R)$  gravity that is more general

than the one introduced in ref. [5], which for the special choice of parameter  $\mu = 0$  coincides with the degenerate model introduced in ref. [15]. Its ultraviolet properties are discussed and it is demonstrated that such  $F(R)$  gravity may be renormalizable for the case  $z = 3$  in a similar way as the original proposal for Hořava-Lifshitz gravity. The Hamiltonian analysis of the proposed modified Hořava-Lifshitz  $F(R)$  gravity shows that this theory is generally consistent with reasonable assumptions. The  $F(R)$  gravity action has also been analyzed in the fixed gauge form, where the presence of the extra scalar is particularly illustrative. The methods presented in the Hamiltonian analyses of sections III and IV can be used to study any action of the general form (5).

The spatially-flat FRW cosmology of the modified Hořava-Lifshitz  $F(R)$  gravity is studied. It is shown that it coincides with the one of the earlier model [5], but only in the spatially-flat FRW case. For specific choice of the parameters of the theory, its FRW equations of motion coincide with the ones of the traditional  $F(R)$  gravity. The presence of the multiple de Sitter solutions shows the principal possibility of the unification of the early-time inflation with the late-time acceleration in the Hořava-Lifshitz background, which proves that it can have rich cosmological applications. The power-law theories are investigated in detail. A number of analytical FRW solutions is found, including the ones with behavior relevant for the early/late cosmic acceleration. The quintessence/phantom-like cosmologies derived in our work may show all the four possible types of finite-time future singularities like in the case of standard dark energy. The conditions to cure such future singularities are discussed in analogy with the traditional  $F(R)$  gravity. It is also interesting that the correction to the Newton law in the  $F(R)$  gravity under discussion can be made unobservably small. Finally, a covariant proposal for  $F(R)$  gravity in Hořava-Lifshitz spirit has been made.

Despite some successes in the formulation of modified Hořava-Lifshitz  $F(R)$  gravity which can be made renormalizable and in its cosmological applications, a number of unsolved questions remain. What is the appropriate way to introduce matter in the theory? Is the theory itself fundamental (or at least, fully consistent) or does it descend from another more fundamental proposal? Can it comply with all the local tests in the Solar system as well as with cosmological bounds? What is the dynamical scenario for the restoration of the Lorentz invariance at late times? What are the cosmological and astrophysical consequences of the first-order modified Hořava-Lifshitz gravity when compared with those of the traditional modified gravity [39]. Moreover, the traditional questions about the properties of black holes in such a theory can be straightforwardly investigated.

Nevertheless, even at the present stage some surprises can be expected from the theory.

While the universe has likely undergone a period of inflation in its early moments, it is interesting to note that Hořava-Lifshitz gravity could produce cosmological perturbations that are almost scale-invariant even without inflation [40].

Hořava-Lifshitz gravity has also been considered in the presence of scalar fields [41, 42]. In principle, it is possible to extend our Hořava-Lifshitz  $F(R)$  gravity by including its coupling with scalar fields.

We would also like to mention a recent paper [43], where a new class of Lorentz-invariance breaking non-relativistic string theories, inspired by the Hořava-Lifshitz gravity, has been presented and analyzed.

Using the  $F(R)$  version of gravity one can propose even a more general formulation of string theory in the Hořava-Lifshitz background: for instance, rigid strings, membranes and  $p$ -branes, etc. On the other hand, it may suggest unusual solutions for the known cosmological problems. There also exists an attempt to explain the homogeneity of our universe in a model with varying speed of light [44]. Having in mind that in the ultraviolet region the speed of the Hořava-Lifshitz graviton changes, one may speculate that the homogeneity of the universe may be described without the need for inflation. In any case, such a theory is both theoretically and cosmologically rich and it deserves further study.

### Acknowledgments

This research has been supported in part by MEC (Spain) project FIS2006-02842 and AGAUR (Catalonia) 2009SGR-994 (SDO), by Global COE Program of Nagoya University (G07) provided by the Ministry of Education, Culture, Sports, Science & Technology (SN). M. O. is supported by the Finnish Cultural Foundation. The support of the Academy of Finland under the Projects No. 121720 and 127626 is gratefully acknowledged.

### Appendix A: Proposal for a covariant $F(R)$ gravity

In [38], a new type of covariant Hořava-Lifshitz-like gravity has been proposed. The action is given, for  $z = 2$  case

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} - \alpha (T^{\mu\nu} R_{\mu\nu} + \beta TR)^2 \right\}, \quad (\text{A1})$$

and for  $z = 3$  case

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} - \alpha (T^{\mu\nu} R_{\mu\nu} + \beta TR) (T^{\mu\nu} \nabla_\mu \nabla_\nu + \gamma T \nabla^\rho \nabla_\rho) (T^{\mu\nu} R_{\mu\nu} + \beta TR) \right\}, \quad (\text{A2})$$

and for  $z = 2n + 2$  case,

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \alpha \{ (T^{\mu\nu} \nabla_\mu \nabla_\nu + \gamma T \nabla^\rho \nabla_\rho)^n (T^{\mu\nu} R_{\mu\nu} + \beta TR) \}^2 \right]. \quad (\text{A3})$$

Here  $T_{\mu\nu}$  is the energy-momentum tensor of the perfect fluid with constant equation of state parameter  $w$  and the parameters  $\beta$  and  $\gamma$  are given by

$$\beta = -\frac{w-1}{2(3w-1)}, \quad \gamma = \frac{1}{3w-1}. \quad (\text{A4})$$

Note that this fluid is not the standard matter fluid. It may have a stringy origin or it may be a kind of gravitational fluid. Hence, we may consider the following covariant Hořava-Lifshitz-like gravity:

$$S_{F(\tilde{R}_{\text{cov}})} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(R_{\text{cov}}),$$

$$R_{\text{cov}} = \begin{cases} R - 2\alpha\kappa^2 (T^{\mu\nu} R_{\mu\nu} + \beta TR)^2, & z = 2, \\ R - 2\kappa^2\alpha (T^{\mu\nu} R_{\mu\nu} + \beta TR) (T^{\mu\nu} \nabla_\mu \nabla_\nu + \gamma T \nabla^\rho \nabla_\rho) (T^{\mu\nu} R_{\mu\nu} + \beta TR), & z = 3, \\ R - 2\kappa^2\alpha (T^{\mu\nu} R_{\mu\nu} + \beta TR) (T^{\mu\nu} \nabla_\mu \nabla_\nu + \gamma T \nabla^\rho \nabla_\rho)^n (T^{\mu\nu} R_{\mu\nu} + \beta TR), & z = 2n + 2. \end{cases} \quad (\text{A5})$$

The theories given by the action (A5) could be renormalizable when  $z \geq 3$ . This may be demonstrated using the same arguments as in ref. [38]. This is a large class of covariant  $F(R)$  gravities whose ultraviolet properties are more similar to the ones of Hořava-Lifshitz-like gravity. However, in many respects their cosmology is similar to the cosmology of traditional modified theories of gravity [1].

- 
- [1] S. Nojiri and S. D. Odintsov, eConf **C0602061**, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007)] [arXiv:hep-th/0601213].
- [2] S. Nojiri and S. D. Odintsov, Phys. Lett. B **576**, 5 (2003) [arXiv:hep-th/0307071].
- [3] P. Hořava, Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775 [hep-th]].
- [4] T. Takahashi and J. Soda, Phys. Rev. Lett. **102**, 231301 (2009) [arXiv:0904.0554 [hep-th]];
- E. Kiritsis and G. Kofinas, Nucl. Phys. B **821**, 467 (2009) [arXiv:0904.1334 [hep-th]];
- R. Brandenberger, Phys. Rev. D **80**, 043516 (2009) [arXiv:0904.2835 [hep-th]];
- S. Mukohyama, K. Nakayama, F. Takahashi and S. Yokoyama, Phys. Lett. B **679**, 6 (2009) [arXiv:0905.0055 [hep-th]];
- T. P. Sotiriou, M. Visser and S. Weinfurtner, JHEP **0910**, 033 (2009) [arXiv:0905.2798 [hep-th]];
- E. N. Saridakis, arXiv:0905.3532 [hep-th];
- M. Minamitsuji, arXiv:0905.3892 [astro-ph.CO];
- G. Calcagni, arXiv:0905.3740 [hep-th];
- A. Wang and Y. Wu, JCAP **0907**, 012 (2009) [arXiv:0905.4117 [hep-th]];
- M. I. Park, JHEP **0909**, 123 (2009) [arXiv:0905.4480 [hep-th]];
- M. Jamil, E. N. Saridakis and M. R. Setare, Phys. Lett. B **679**, 172 (2009) [arXiv:0906.2847 [hep-th]];
- M. I. Park, JCAP **1001**, 001 (2010) [arXiv:0906.4275 [hep-th]];
- C. Bogdanos and E. N. Saridakis, arXiv:0907.1636 [hep-th];
- C. G. Boehmer and F. S. N. Lobo, arXiv:0909.3986 [gr-qc];
- I. Bakas, F. Bourliot, D. Lust and M. Petropoulos, arXiv:0911.2665 [hep-th];
- G. Calcagni, JHEP **0909**, 112 (2009) [arXiv:0904.0829 [hep-th]].

- [5] M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen and A. Tureanu, arXiv:1001.4102 [hep-th]; to appear in Class. Quant. Grav.
- [6] R. L. Arnowitt, S. Deser and C. W. Misner, arXiv:gr-qc/0405109, originally “Gravitation: An Introduction to Current Research”, L. Witten ed., Wiley, New York, 1962;  
C. Gao, arXiv:0905.0310 [astro-ph.CO].
- [7] R. M. Wald, *General Relativity*, University of Chicago Press, 1984, Chicago and London.  
É.ourgoulhon, arXiv:gr-qc/0703035.
- [8] R. P. Woodard, Lect. Notes Phys. **720**, 403 (2007) [arXiv:astro-ph/0601672].
- [9] M. Henneaux, A. Kleinschmidt and G. L. Gómez, arXiv:0912.0399 [hep-th].  
M. Li and Y. Pang, JHEP **0908**, 015 (2009) [arXiv:0905.2751 [hep-th]].
- [10] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfar Graduate School of Science, Yeshiva University, 1964, New York. Reprinted by Dover Publications, Mineola, New York, in 2001.
- [11] M. Chaichian and N. F. Nelipa, *Introduction to Gauge Field Theories*, Springer-Verlag, 1984, Berlin and Heidelberg;  
M. Chaichian and A. Demichev, *Path Integrals in Physics*, Vol. I and II, Institute of Physics Publishing, 2001, Bristol and Philadelphia.
- [12] D. M. Gitman and I. V. Tyutin. *Quantization of Fields with Constraints*, Springer-Verlag, 1990, Berlin and Heidelberg.
- [13] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, 1994, Princeton, New Jersey.
- [14] J. Klusoň, arXiv:1002.4859 [hep-th].
- [15] J. Klusoň, arXiv:0910.5852 [hep-th].
- [16] A. Cabo, M. Chaichian and D. L. Martinez. J. Math. Phys. **34** (1993) 5646
- [17] M. Chaichian, D. L. Martinez and L. Lusanna, Annals Phys. **232** (1994) 40.
- [18] S. Carloni, E. Elizalde and P. J. Silva, Class. Quantum Grav. **27** (2010) 045004 [arXiv:0909.2219 [hep-th]].
- [19] S. Mukohyama, Phys. Rev. D **80**, 064005 (2009) [arXiv:0905.3563 [hep-th]].
- [20] S. Nojiri, S. D. Odintsov and D. Saez-Gomez, Phys. Lett. B **681**, 74 (2009) [arXiv:0908.1269 [hep-th]].
- [21] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003) [arXiv:hep-th/0307288].
- [22] S. Nojiri and S. D. Odintsov, Phys. Rev. D **77**, 026007 (2008) [arXiv:0710.1738 [hep-th]].
- [23] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **77**, 046009 (2008) [arXiv:0712.4017 [hep-th]].
- [24] G. Cognola, E. Elizalde, S. D. Odintsov, P. Tretyakov and S. Zerbini, Phys. Rev. D **79**, 044001 (2009) [arXiv:0810.4989 [gr-qc]].
- [25] S. Carloni, A. Troisi and P. K. S. Dunsby, Gen. Rel. Grav. **41** (2009) 1757 [arXiv:0706.0452 [gr-qc]].
- [26] S. Carloni, P. K. S. Dunsby, S. Capozziello and A. Troisi, Class. Quant. Grav. **22** (2005) 4839 [arXiv:gr-qc/0410046].
- [27] S. Nojiri, S. D. Odintsov, A. Toporensky and P. Tretyakov, arXiv:0912.2488 [hep-th].
- [28] G. Cognola, E. Elizalde, S. Nojiri and S. D. Odintsov, arXiv:0909.2747 [gr-qc].
- [29] S. Nojiri and S. D. Odintsov, J. Phys. Conf. Ser. **66** (2007) 012005 [arXiv:hep-th/0611071];  
S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006) [arXiv:hep-th/0608008].
- [30] S. Nojiri, S. D. Odintsov and M. Sami, Phys. Rev. D **74** (2006) 046004 [arXiv:hep-th/0605039].
- [31] E. Elizalde, R. Myrzakulov, V. V. Obukhov and D. Saez-Gomez, arXiv:1001.3636 [gr-qc].
- [32] R. R. Caldwell, Phys. Lett. B **545**, 23 (2002) [arXiv:astro-ph/9908168];  
R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003) [arXiv:astro-ph/0302506];  
B. McInnes, JHEP **0208** (2002) 029 [arXiv:hep-th/0112066];  
S. Nojiri and S. D. Odintsov, Phys. Lett. B **562**, 147 (2003) [arXiv:hep-th/0303117];  
S. Nojiri and S. D. Odintsov, Phys. Lett. B **571**, 1 (2003) [arXiv:hep-th/0306212];  
V. Faraoni, Int. J. Mod. Phys. D **11**, 471 (2002) [arXiv:astro-ph/0110067];  
P. F. Gonzalez-Diaz, Phys. Lett. B **586**, 1 (2004) [arXiv:astro-ph/0312579]; TSPU Vestnik **44N7**, 36 (2004) [arXiv:hep-th/0408225];  
M. Sami and A. Toporensky, Mod. Phys. Lett. A **19**, 1509 (2004) [arXiv:gr-qc/0312009];  
H. Stefancic, Phys. Lett. B **586**, 5 (2004) [arXiv:astro-ph/0310904];  
L. P. Chimento and R. Lazkoz, Phys. Rev. Lett. **91**, 211301 (2003) [arXiv:gr-qc/0307111];  
Mod. Phys. Lett. A **19**, 2479 (2004) [arXiv:gr-qc/0405020];  
J. G. Hao and X. Z. Li, Phys. Lett. B **606**, 7 (2005) [arXiv:astro-ph/0404154];  
E. Babichev, V. Dokuchaev and Yu. Eroshenko, Class. Quant. Grav. **22**, 143 (2005) [arXiv:astro-ph/0407190];  
X. F. Zhang, H. Li, Y. S. Piao and X. M. Zhang, Mod. Phys. Lett. A **21**, 231 (2006) [arXiv:astro-ph/0501652];  
E. Elizalde, S. Nojiri, S. D. Odintsov and P. Wang, Phys. Rev. D **71**, 103504 (2005) [arXiv:hep-th/0502082];  
M. P. Dabrowski and T. Stachowiak, Annals Phys. **321**, 771 (2006) [arXiv:hep-th/0411199];  
F. S. N. Lobo, Phys. Rev. D **71**, 084011 (2005) [arXiv:gr-qc/0502099];  
R. G. Cai, H. S. Zhang and A. Wang, Commun. Theor. Phys. **44**, 948 (2005) [arXiv:hep-th/0505186];  
I. Y. Aref’eva, A. S. Koshelev and S. Y. Vernov, Theor. Math. Phys. **148**, 895 (2006) [arXiv:astro-ph/0412619]; Phys. Rev.

- D **72**, 064017 (2005) [arXiv:astro-ph/0507067];  
H. Q. Lu, Z. G. Huang and W. Fang, arXiv:hep-th/0504038;  
W. Godlowski and M. Szydlowski, Phys. Lett. B **623**, 10 (2005) [arXiv:astro-ph/0507322];  
J. Sola and H. Stefancic, Phys. Lett. B **624**, 147 (2005) [arXiv:astro-ph/0505133];  
B. Guberina, R. Horvat and H. Nikolic, Phys. Rev. D **72**, 125011 (2005) [arXiv:astro-ph/0507666];  
M. P. Dabrowski, C. Kiefer and B. Sandhofer, Phys. Rev. D **74**, 044022 (2006) [arXiv:hep-th/0605229];  
E. M. Barbaoza and N. A. Lemos, arXiv:gr-qc/0606084.
- [33] J. D. Barrow, Class. Quant. Grav. **21**, L79 (2004) [arXiv:gr-qc/0403084];  
S. Cotsakis and I. Klaoudatou, J. Geom. Phys. **55**, 306 (2005) [arXiv:gr-qc/0409022];  
S. Nojiri and S. D. Odintsov, Phys. Lett. B **595**, 1 (2004) [arXiv:hep-th/0405078]; Phys. Rev. D **70**, 103522 (2004) [arXiv:hep-th/0408170];  
J. D. Barrow and C. G. Tsagas, Class. Quant. Grav. **22**, 1563 (2005) [arXiv:gr-qc/0411045];  
M. P. Dabrowski, Phys. Rev. D **71**, 103505 (2005) [arXiv:gr-qc/0410033]; A. Balcerzak and M. P. Dabrowski, Phys. Rev. D **73**, 101301 (2006) [arXiv:hep-th/0604034];  
L. Fernandez-Jambrina and R. Lazkoz, Phys. Rev. D **70**, 121503 (2004) [arXiv:gr-qc/0410124]; Phys. Rev. D **74**, 064030 (2006) [arXiv:gr-qc/0607073]; arXiv:0805.2284 [gr-qc];  
S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005) [arXiv:hep-th/0505215];  
P. Tretyakov, A. Toporensky, Y. Shtanov and V. Sahni, Class. Quant. Grav. **23**, 3259 (2006) [arXiv:gr-qc/0510104];  
H. Stefancic, Phys. Rev. D **71**, 084024 (2005) [arXiv:astro-ph/0411630];  
A. V. Yurov, A. V. Astashenok and P. F. Gonzalez-Diaz, Grav. Cosmol. **14**, 205 (2008) [arXiv:0705.4108 [astro-ph]];  
I. Brevik and O. Gorbunova, Eur. Phys. J. C **56**, 425 (2008) [arXiv:0806.1399 [gr-qc]];  
M. Bouhmadi-Lopez, P. F. Gonzalez-Diaz and P. Martin-Moruno, Phys. Lett. B **659**, 1 (2008) [arXiv:gr-qc/0612135]; arXiv:0707.2390 [gr-qc];  
M. Sami, P. Singh and S. Tsujikawa, Phys. Rev. D **74**, 043514 (2006) [arXiv:gr-qc/0605113];  
C. Cattoen and M. Visser, Class. Quant. Grav. **22**, 4913 (2005) [arXiv:gr-qc/0508045];  
J. D. Barrow and S. Z. W. Lip, arXiv:0901.1626 [gr-qc];  
T. Koivisto, Phys. Rev. D **77**, 123513 (2008) [arXiv:0803.3399 [gr-qc]];  
M. Bouhmadi-Lopez, Y. Tavakoli and P. V. Moniz, arXiv:0911.1428 [gr-qc].
- [34] S. Nojiri and S. D. Odintsov, Phys. Rev. D **78**, 046006 (2008) [arXiv:0804.3519 [hep-th]];  
K. Bamba, S. Nojiri and S. D. Odintsov, JCAP **0810**, 045 (2008) [arXiv:0807.2575 [hep-th]];  
S. Capozziello, M. De Laurentis, S. Nojiri and S. D. Odintsov, arXiv:0903.2753 [hep-th];  
K. Bamba, S.D. Odintsov, L. Sebastiani and S. Zerbini, arXiv:0911.4390.
- [35] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71** 063004 (2005) [arXiv:hep-th/0501025]  
[36] M. C. B. Abdalla, S. Nojiri and S. D. Odintsov, Class. Quant. Grav. **22**, L35 (2005) [arXiv:hep-th/0409177].  
[37] E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004) [arXiv:hep-th/0405034].  
[38] S. Nojiri and S. D. Odintsov, arXiv:0905.4213 [hep-th].  
[39] S. Capozziello and M. Francaviglia, Gen. Relat. Grav. **40**, 357 (2008) [arXiv:0706.1146 [astro-ph]];  
S. Capozziello, M. de Laurentis and V. Faraoni, arXiv:0909.4672;  
F. Schmidt, A. Vikhlinin and W. Hu, Phys. Rev. D **80**, 083505 (2009) [arXiv:0908.2457 [astro-ph.CO]];  
Y.-S. Song, H. Peiris and W. Hu, Phys. Rev. D **76**, 063517 (2007) [arXiv:0706.2399 [astro-ph]].
- [40] S. Mukohyama, JCAP **0906**, 001 (2009) [arXiv:0904.2190 [hep-th]].  
[41] B. Chen and Q. G. Huang, Phys. Lett. B **683**, 108 (2010) [arXiv:0904.4565 [hep-th]].  
[42] J. Lee, T. H. Lee and P. Oh, arXiv:1003.2840 [hep-th].  
[43] J. Klusoň, arXiv:1002.2849 [hep-th].  
[44] A. J. Albrecht and J. Magueijo, Phys. Rev. D **59**, 043516 (1999) [arXiv:astro-ph/9811018].